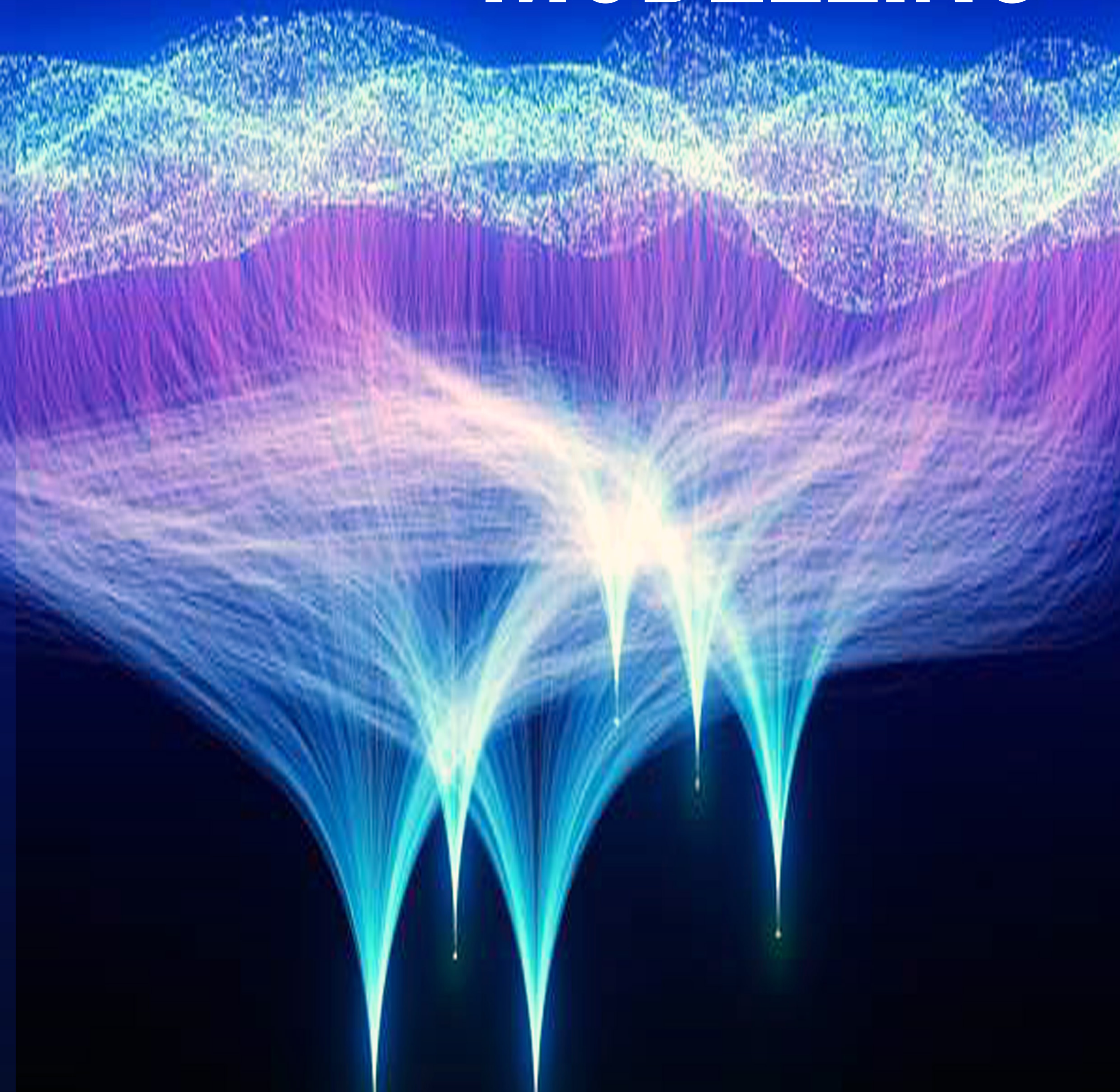
MATHEMATICAL MODELLING



DR.R.THAYALARAJAN

மனோன்மணியம் சுந்தரனார் பல்கலைக்கழகம்

MANONMANIAM SUNDARANAR UNIVERSITY

TIRUNELVELI – 627 012

தொலைநிலை தொடர் கல்வி இயக்ககம்

DIRECTORATE OF DISTANCE AND

CONTINUING EDUCATION

B.Sc., MATHEMATICS

III YEAR

MATHEMATICAL MODELLING

Sub. Code: JMMA53

Prepared by

Dr. R. THAYALARAJAN

Assistant Professor

Department of Mathematics

M.S. University College, Govindaperi – 627 414.

II YEAR - B.SC., MATHEMATICS

MATHEMATICAL MODELLING

SYLLABUS

Unit I:

Mathematical Modelling: Simple situation requiring mathematical modelling, characteristics of mathematical models.

Chapter 1: Sec 1.1 and 1.4

Unit II:

Mathematical Modelling through differential equations: Linear Growth and Decay Models. Non-Linear growth and decay models, compartment models.

Chapter 2: Sec 2.1 to 2.4

Unit III:

Mathematical Modelling through system of ordinary differential equations of first order: Prey-predator models, competition models, Model with removal and model with immigrations. Epidemics: simple epidemic model, Susceptible – infected – susceptible (SIS) model, SIS model with constant number of carriers. Medicine: Modelfor Diabetes Mellitus.

Chapter 3: Sec 3.1.1, 3.1.2, 3.2.1 to 3.2.4, 3.2.6, 3.5.1

Unit IV:

Introduction to difference equations.

Chapter 5: Sec 5.1, 5.2.1 to 5.2.3

Unit V:

Mathematical Modelling through difference equations: Harrod Model, cob web model application to Actuarial Science.

Chapter 5: Sec 5.3.1, 5.3.2, 5.3.4

Text Book:

1.J.N.Kapur, Mathematical Modeling, New Age International Publishers, 2009.

UNIT – I
MATHEMATICAL MODELLING

Introduction:

Models describe our beliefs about how the world functions. In mathematical modelling, we translate those beliefs into the language of mathematics. Mathematical modeling is the process of using mathematics to describe, simulate, and analyze real-world systems and problems. It involves translating a real-world situation into a mathematical representation, often using equations and graphs, to understand its behavior, make predictions, and find solutions. This process helps bridge the gap between abstract theory and practical application in fields like science, engineering, and economics. This has many advantages

1. Mathematics is a very precise language. This helps us to formulate ideas and identify underlying assumptions.
2. Mathematics is a concise language, with well-defined rules for manipulations.
3. All the results that mathematicians have proved over hundreds of years are at our disposal.
4. Computers can be used to perform numerical calculations.

Objectives:

Mathematical modelling can be used for a number of different reasons. How well any particular objective is achieved depends on both the state of knowledge about a system and how well the modelling is done. For example

1. Developing scientific understanding - through quantitative expression of current knowledge of a system (as well as displaying what we know, this may also show up what we do not know)
2. test the effect of changes in a system
3. aid decision making, including

   (i) tactical decisions by managers (ii) strategic decisions by planners.

Needs of Mathematical Modelling:

Mathematical modeling is needed to understand, predict, and manipulate complex real-world systems by simplifying them into a mathematical framework. Key needs include making informed decisions, improving resource efficiency by avoiding costly physical prototypes, and fostering innovation through systematic problem-solving. It is a fundamental tool across science, engineering, medicine, economics, and more for simulating scenarios and creating technology.

1. Understanding and simplification:

It provides a systematic framework to understand and analyze complex phenomena that are too intricate for intuition alone. By creating a simplified mathematical representation, we can focus on key relationships and variables.

2. Prediction and forecasting:

Mathematical models are used to predict future scenarios, such as weather forecasts, the spread of diseases, or economic trends.

3. Informed decision-making:

By quantifying relationships and dependencies, models provide a basis for better decisions in fields like engineering, public health, and business.

4. Efficiency and cost reduction:

Modelling allows for the simulation and testing of systems on computers, which is often faster and less expensive than developing and testing physical prototypes.

5. Innovation and design:

Mathematical modelling is crucial for innovation, providing a platform to explore new ideas and design new technologies, from computer graphics to new engineering designs.

6. Problem-solving:

It offers a method for tackling real-world challenges by translating a problem into mathematical terms, solving the resulting equations, and then translating the solution back to the real world.

SIMPLE SITUATIONS REQUIRING MATHEMATICAL MODELLING:

Consider the following problems:

(i) Find the height of a tower, say the Washington Monument or the leaning tower at Pisa (without climbing it!).

(ii) Find the width of a river or a canal (without crossing it!).

(iii) Find the mass of the Earth (without using a balance!).

(iv) Find the temperature at the surface or at the centre of the Sun (without taking a thermometer there!).

(v) Estimate the yield of wheat in India from the standing crop (without cutting and weight the whole of it!).

(vi) Find the volume of blood inside the boby of the human begin (without bleeding him to death!).

(vii) Estimate the population of china in the year 2050 A.D. (without waiting till then!).

(viii) Find the time it takes a satellite at a height of 10,000 kms above the earth's surface to complete one orbit (without sending such a satellite into orbit!).

(ix) Find the effect on the economy of a 30% reduction in income tax (without actually reduction the rate!).

(x) Find the gun with the performance when the performance depends on ten parameters, each of which can take ten values (without manufacturing 10^{10} guns!)

(xi) Estimate the average life span of a light bulb manufactured in a factory (without lighting each bulb till it gets fused!).

(xii) Estimate the total amount of insurance claims a company has to pay next year (without waiting till end of the year!).

All these problems and thousands of similar problems can be and have been solved through mathematical modeling.

One technique of solving the previous problems is similar to that of solving "world problems" in algebra. Suppose the age of a father is four times the age of his son and we are told that after five years, the age of the father will be only three times the age of the son. We have to find their ages. Let x be the age of the father and y be the age of the son, then the data of the problem gives

$x = 4y$ -------------- (1)

$x + 5 = 3(y + 5)$ --------------- (2)

By solving (1) and (2) we have $x = 40, y = 10$.

The two equation (1) and (2) give a mathematical model of the biological situation, so that the biological problem of ages is reduced to the mathematical problem of the solution of a system of two algebraic equations. The solution of the equations is finally interpreted biologically to give the ages of the father and the son.

The mathematical relations we get may be in terms of algebraic, transcendental, differential, difference, integral, integro-differential, differential-difference equations, or even in terms of inequalities. Thus, For

(i) We try to express the height of the tower in terms of some distances and angles which can be measured on the ground.

(ii) We try to express the width of the tower in terms of some distances and angles which can be measured on our side of the river.

(iii) We try to express the mass of the Earth in terms of some known masses and distances.

(iv) We try to express the temperatures at the surface and the centre of the Sun in terms of the properties of light received from its surface.

(v) We try to find the area under wheat and the average yield per acre by cutting and weighting the crop from some representative plots.

(vi) We inject some glucose into the blood stream and find the increase in the concentration of sugar in the blood.

(vii) We extrapolated from date from previous censuses or develop a model expressing the population as a function of time.

(viii) We try to use Newton's laws to get a relation between the orbital period and the height of the satellite above the surface of the Earth.

(ix) We examine the effects of similar cuts in the past or develop a mathematical model giving the relation between income-tax cuts, purchasing power in the hands of individuals and its effects on productivity and inflation, etc.

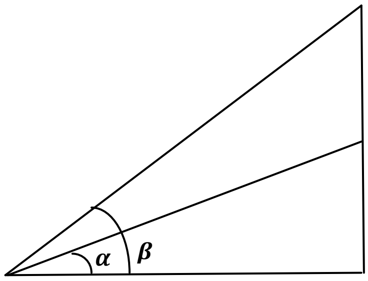
(x) We develop a theory of internal ballistics of guns based on laws of burning of propellants, motion of gases inside a gun and motion of the shot inside it.

(xi) We take a random sample of bulbs, find their life span and use statistical inference models to estimate the life span for the population of bulbs.

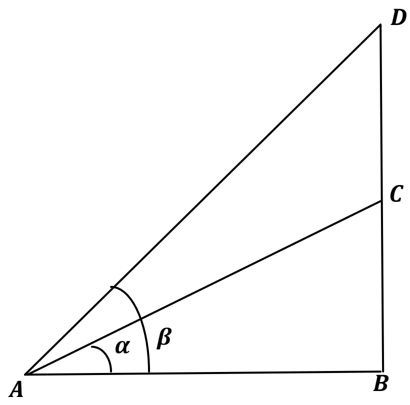
(xii) We use probabilistic models for life expectancy of individuals.

PROBLEMS

1. The angle of elevation of the foot and the top of a flagpole on a tower, from a point $\boldsymbol{a}$ meters from the foot of the tower $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ respectively in following figure. Show that the height of the flagpole is $\boldsymbol{a(tan\,\beta - tan\,\alpha)}$ meters.



Solution:



Let A be the observation point, a meters from the foot of the tower. Let B be the foot of the tower and C be the top of the tower. Let D be the top of the flagpole, which is on the tower.

Let height of the tower is $BC = h_1$ and height of the flagpole is $CD = h_2$. The total height from the ground to the top of the flagpole is $BD = h_1 + h_2$. The distance from the observation point to the foot of the tower is $AB = a$.

To Prove: The height of the flagpole is $a(\tan\beta - \tan\alpha)$

From the point A:

The angle of elevation to the top of the tower is α.

In ΔABC, we have $\tan\alpha = \frac{BC}{AB} = \frac{h_1}{a}$

Therefore, $h_1 = a\tan\alpha$ ------------------- (1)

From the same point A:

The angle of elevation to the top of the flagpole is β.

In ΔABD, we have $\tan\beta = \frac{BD}{AB} = \frac{h_1+h_2}{a}$

Therefore, $h_1 + h_2 = a\tan\beta$ ------------- (2)

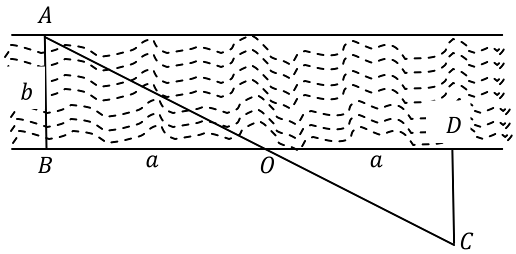
Substituting (2) in (1), we have

$a\tan\alpha + h_2 = a\tan\beta$

$h_2 = a\tan\beta - a\tan\alpha$

$h_2 = a(\tan\beta - \tan\alpha)$ meters.

2. Explain how you would find the breath of a river without crossing it (use Figure)



Solution:

To find DC:

Let $AB \perp BD$ and $DC \perp BD$

Let O be the mid-point of BD.

$\Rightarrow BO = OD$

In ΔAOB and ΔCOD,

$\angle ABO = \angle CDO$ (each angle is 90^0)

$BO = DO$ (O is the mid-point of BD)

Here the angle $\angle AOB = \angle COD$ is vertically opposites

By $\Delta AOB \cong \Delta COD$. Hence, we know that corresponding parts of the congruent triangles are equal. Therefore, $AB = DC = b$.

3. Show that of all rectangles with a given perimeter, the square has the maximum area. Also, Show that of all rectangles with a given area, the square has the minimum perimeter.

Solution:

Let us consider the rectangle have length x and breath y.

W.k.t. the perimeter of the rectangle $P = 2(l + b)$

$P = 2(x + y)$ ---------------------- (A)

$\frac{P}{2} = x + y \Rightarrow y = \frac{P}{2} - x$ ---------------------- (1)

W.k.t. the area of the rectangle $A = lb$

$A = xy$ --------------------- (B)

Substitute the (B) in (1), we have

$A = x\left(\frac{P}{2} - x\right)$

$= \frac{P}{2}x - x^2$ ---------------------- (2)

To find maximum area:

Differentiating (2) w.r.to $'x'$, we get

$\frac{dA}{dx} = \frac{P}{2} - 2x$

When $\frac{dA}{dx} = 0$, we have $x = \frac{P}{4}$

Substitute the value of x in (1), we have $y = \frac{P}{4}$

Therefore, $x = \frac{P}{4}$ and $y = \frac{P}{4}$ is the rectangle with the maximum area for a given perimeter is a square.

To find minimum perimeter:

(B) $\Longrightarrow y = \frac{A}{x}$

Substitute the (A) in (1), we have

$$P = 2\left(x + \frac{A}{x}\right)$$

$$= 2x + \frac{2A}{x}$$

The above equation differentiating w.r. to $'x'$, we have

$$\frac{dP}{dx} = 2 - \frac{2A}{x^2}$$

When $\frac{dP}{dx} = 0$, we have $2 - \frac{2A}{x^2} = 0$

$$x^2 = A \Longrightarrow x = \sqrt{A}$$

Substitute the value of x in (1), we have $y = \frac{A}{\sqrt{A}} = \sqrt{A}$

Therefore, $x = \sqrt{A}$ and $y = \sqrt{A}$ is the rectangle with the minimum perimeter for the given area is a square.

4. Let A and B be two places d miles apart on the surface of the earth and having the same longtitude and with latitude θ_1 and θ_2 degree respectively. show that the radius of the earth in miles is given by $a = d\left(\frac{180}{\pi(\theta_2 - \theta_1)}\right)$.

Solution:

Let the distance d between two points on the surface of the Earth with the same longitude is the arc length along a great circle. This arc length is given by the $s = r\theta$.

Where s is the arc length, r be the radius of the circle and θ is the central angle in radians. In this case, the arc length s is the distance d between places A and B and the radius r is the radius of the earth denoted by a.

The central angle between the two places is the difference in their latitudes, which is $\theta_2 - \theta_1$ degrees.

To use the arc length formula, we must convert the angle from degrees to radians.

$$\text{(i.e) } \theta_{radians} = \theta_{degree}\left(\frac{\pi}{180}\right)$$

Therefore, the central angle in radians is) $\theta_{radians} = (\theta_2 - \theta_1)\left(\frac{\pi}{180}\right)$

Substitute the value for arc length d and the central angle in radians into the arc length

$$d = a \times (\theta_2 - \theta_1)\left(\frac{\pi}{180}\right)$$

$$a = d \times \frac{180}{\pi(\theta_2 - \theta_1)}.$$

5. 5mgs of glucose are introduced in to the blood stream and after 2minutes, a samples of 10ccs of blood is taken in which the increase in blood sugar is found to be $\mathbf{0.01}$mg. Estimate the volume of blood in the body.

Solution:

The concentration of glucose increase in the blood sample is calculated by by dividing the increase in mass by the volume of the sample.

i) increase in mass $m_1 = 0.01$mg

ii) volume of sample $V_1 = 10$ccs or 10ml

$$\text{Concentration } C = \frac{m_1}{V_1} = \frac{0.01}{10} = 0.001\text{mg/ml}$$

To estimate the volume of blood:

The total amount of glucose introduced (5mg) is distributed throughout the entire volume of blood in the body. The concentration increase found in the sample is assumed to be the same as the concentration increase in the entire blood volume. Given, the total mass of glucose introduced $m_2 = 5$mg

$$\text{Volume of blood } V = \frac{m_2}{C} = \frac{5\text{mg}}{0.001\text{mg/ml}} = 5000\text{ml or 5 litres.}$$

6. A random sample of 100 light bulbs is found to have a mean life span of 200 hours and a SD of 10 hours. What statement can you make about bulbs made in the factory?

Solution:

Here the sample size $n = 100$, sample mean $\bar{x} = 200$, SD $\sigma = 10$.

Now, test of significance level for a 95% confidence interval.

(i.e) $\alpha = 1 - 0.95 = 0.05$

and since the sample size $(n = 100)$ is large (typically ≥ 30), we can use the Z-distribution. For a 95% confidence interval then, the critical value $Z_{\frac{\alpha}{2}} = 1.96$.

$$\text{Standard Error: S.E} = \frac{\sigma}{\sqrt{n}} = \frac{10}{\sqrt{100}} = 1\text{hour.}$$

The maximum likely difference between the sample mean and the population mean.

$$\mathrm{ME} = \mathrm{Z}_{\frac{\alpha}{2}} \times \mathrm{S.E} = 1.96 \times 1 = 1.96 \text{ hours.}$$

Therefore, confidence interval $\mathrm{CI} = \bar{x} \pm \mathrm{ME} = 200 \pm 1.96$

Since, we can state with 95% confidence that the true mean lifespan of all bulbs produced in the factory is between 198.04 and 201.96 hours.

7. Explain how you would find the volume of water in a village pond.

Solution:

The importance of getting an accurate estimation of your pond surface area cannot be overestimated. The majority of pond owners visually estimate their pond area, which usually results in an overestimate of the true pond surface area. Pond area and water volume should be calculated based on some simple measurements. The effort necessary to estimate pond surface area is directly related to your pond's shape and uniformity. The simplest method using basic equations for common shapes can be applied if your pond closely resembles a circle, square, rectangle, or trapezoid in shape.

In Circle: Circular pond shape can be estimated by measuring the distance around the pond shoreline in feet.

Volume of the circular shape like cylinder $= \pi r^2 h$

In rectangle: Rectangle pond shape can be estimated by measuring the distance around the pond shoreline in feet.

Volume of the rectangular box $= lbh$

In Trapezoid: Many ponds may be roughly rectangular in shape, but one side may be significantly shorter than the other. The area of this shape is best estimated using a formula for a trapezoid by taking the average length of the two unequal sides and multiplying by the width of the pond.

Volume of the trapezoid $= \frac{1}{2}(a + b)hl$, where a and b are the lengths of the parallel sides of the trapezoid.

Irregular Shaped: Many ponds have an irregular shape where the surface area cannot be adequately estimated using the formulas for common geometric shapes. Three methods can be used in this case, depending on the degree of accuracy you desire. Keep in mind that the accuracy of your pond surface area estimate may be very important, especially for the safe use of aquatic herbicides. The three methods are described in order from least to most accurate. You should strive to use the most accurate method that you can reasonably accomplish.

1. Average Length and Width Method: Take numerous measurements to determine the average length and average width. Make certain you get both the longest and shortest distances in calculating the average length, and the widest and narrowest distances for determining the average width. The more measurements that you make, the more accurate your result will be. The area is then calculated by multiplying the average width by the average length.

2. Multiple Trapezoids Method: A more accurate method to determine the area of an odd-shaped pond is to divide the pond into multiple trapezoid shapes. A new trapezoid is defined anywhere the shoreline makes a rapid change in direction. Note that instead of horizontal transects, this method requires measurement of the distance between each vertical transect. This would be most easily done during winter when the pond is frozen and the transects could be easily laid out and measured. This method requires more measurement and effort, but the final area estimate will probably be within ±5 to 10 percent of the actual pond area.

3. Handheld Global Positioning Systems (GPS): Handheld GPS systems have become quite common over the past ten years as they have become more affordable. They are now routinely used for outdoor recreation (hunting, hiking, camping, etc.) and navigation. GPS units allow you to determine your exact location on Earth using multiple satellites in space. Various locations, or "waypoints," can be stored in the GPS unit for use with mapping software that either accompanies the unit or can be purchased separately. A pond surface area could be estimated by walking the perimeter of the pond and stopping at various waypoint locations along the pond shoreline. The software can connect the waypoints and calculate the area inside the resulting shape. If waypoints are stored at each location where the pond shape changes, the resulting area will be extremely accurate, probably within 1 percent of the actual pond area. Even if you do not own a GPS system, friends or family members who enjoy outdoor recreation may own a unit that could be used to estimate your pond surface area.

4. Geographical Information Websites: There are also geographical information programs on the internet, like Google Earth or Bing Maps, which use satellite imagery to display a map of your pond or lake. These website tools can make it very easy to determine the surface area of your water resource by tracing the perimeter of your pond.

8. Suggest some methods estimating the heights of mountain peaks and depths of ocean beds.

Solution:

Various methods exist for estimating the heights of mountain peaks and the depths of ocean beds, ranging from traditional surveying techniques to modern remote-sensing technologies. The most accurate modern methods use satellite-based systems.

1. Global Positioning System (GPS): Scientists hike to the summit with a GPS receiver, which collects data over several hours to calculate its location and height relative to an ellipsoid—a mathematical model of the Earth. The data is then corrected to determine the height above mean sea level, considering local gravitational anomalies that affect the measurement.

2. Light Detection and Ranging (LiDAR): An aircraft or satellite equipped with a laser scanner fires pulses of light at the Earth's surface. By measuring the time it takes for the reflected light to return, the system calculates the distance and generates a "point cloud" of 3D points. Specialized software then processes this data to create a high-resolution Digital Elevation Model (DEM), or a "bare earth" model that strips away vegetation to reveal the true ground elevation.

3. Trigonometric Surveying (Triangulation): Surveyors measure a baseline on level ground and use a theodolite to measure the angle of elevation from each end of the baseline to the mountain's peak. Using trigonometry, they can then calculate the mountain's height. This process, known as triangulation, was famously used in the Great Trigonometrical Survey of India.

4. Barometric Altimetry: A barometer is used to measure the air pressure at the base of the mountain and again at the summit. The difference in pressure is used to calculate the altitude change. However, this method is less precise than others because air pressure can also be affected by weather.

5. Photogrammetry: Specialized software analyzes overlapping photos to generate detailed topographical maps and elevation data. This is particularly useful for mapping hard-to-reach areas.

9. Discuss the mathematical bases of the methods used by civil engineers in land surveys.

Solution:

Civil engineers use trigonometry, algebra, geometry, and calculus as the mathematical basis for land surveying to measure distances, angles, elevations, and areas. These methods allow them to locate points, establish property boundaries, calculate land area and volume, and create maps for projects like roads, buildings, and other infrastructure. Key principles include using the Pythagorean theorem for distances and trigonometric functions for angles and elevations.

(i) Trigonometry and geometry:

Distance and elevation: Trigonometry is used to calculate unknown distances and elevations from measured angles and distances. For example, in trigonometric leveling, the height of a point is determined using the horizontal distance and the vertical angle of elevation or depression.

Angles: The sexagesimal system ($360^0, 60'$ per degree, $60''$ per minute) is used for angular measurement. This is essential for determining the direction of lines (azimuths or bearings), often using the arctangent function from coordinates.

Coordinate geometry: The Pythagorean theorem ($= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is fundamental for calculating the distance between two points given their coordinates.

(ii) Algebra:

Area and volume: Basic algebra is used to apply formulas for calculating the area of land, which is crucial for planning and cost estimation.

Traverse methods: The latitude and meridian distance (DMD) method uses latitude and meridian distances from a reference north-south axis to calculate the area of a traverse, with positive and negative signs indicating the direction.

(iii) Calculus and advanced methods:

Area calculation: Calculus-based rules, such as the Trapezoidal Rule and Simpson's Rule, are used for more precise area calculations from irregular boundaries, which are often defined by a series of offsets.

Volume calculation: Calculus is also used to calculate volumes for cut-and-fill operations in construction projects by using cross-sectional areas along a surveyed line.

Advanced principles: Modern surveying applications like geodesy and map projections require more advanced mathematical concepts, including linear algebra, advanced calculus, and statistics, to account for the curvature of the Earth and create accurate geospatial networks.

SOME CHARACTERISTICS OF MATHEMATICAL MODELS:

Mathematical models are characteristics of being abstract and represent real-world situations using mathematical expressions like equations and variables. They often include governing equations, parameters, assumptions, and constraints. Other key characteristics include the ability to make predictions, a hierarchy of models for increased realism, and the distinction between linear/nonlinear and static/dynamic types.

(i) Realism of models: We want a mathematical model to be as realistic as possible and to represent reality as closely as possible. However, if a model is very realistic, it may not be mathematically tractable. In making a mathematical model, there has to be a trade-off between tractability and reality.

(ii) Hierarchy of models: Mathematical modelling is not a one-shot affair. Models are constantly improved to make them more realistic. Thus for every situation, we get a hierarchy of models, each more realistic than the preceding and each likely to be followed by a better one.

(iii) Relative precision of models: Different models differ in their precision and their agreement with observation.

(iv) Robustness of models: A mathematical model is said to be robust if small changes in the parameters lead to small changes in the behaviour of the model. The decision is made by using sensitivity analysis for the models.

(v) Self-consistency of models: A mathematical model involves equations and inequalities and these must be consistent. e.g., a model cannot have both $x + y > a$ and $x + y < a$. Sometimes the inconsistency results from inconsistency of basic assumptions. Since, mathematical inconsistency is relatively easier to find out, this gives a method of finding inconsistency in requirements which social or biological scientists may require of their models. A well-known example of this is provided by Arrow's impossibility theorem.

(vi) Oversimplified and overambitious models: It has been said that mathematics that is certain does not refer to reality and mathematics that refers to reality is not certain. A model may not represent reality because it is oversimplified. A model may also be overambitious in the sense that it may involve too many complications and may give results accurate to ten decimal places. Whereas the observations may be correct to two decimal places only.

(vii) Complexity of models: This can be increase by subdividing variables, by taking more variables and by considering more details. Increase of complexity need not always lead to increase of insight as after a stage, diminishing returns begin to set in. The art of mathematical modelling consists in stopping before this stage.

(viii) Models can lead to new experiments, new concepts and new mathematics: Comparison of predictions with observations reveals the need for new experiments to collect needed data. Mathematical models can also lead to development of new concepts. If known mathematical techniques are not adequate to deduce results from the mathematical model, new mathematical techniques have to be delveloped.

(ix) A model may be good, adequate, similar to reality for one purpose and nor for another: Thus we may need different models for explaining different aspects of the same situation or even for different ranges of the variables. Of course in this case, search for a unified model continues.

(x) Models may lead to expected or unexpected predictions or even to nonsense: Usually models give predictions expected on common sense considerations, but the model predictions are more quantitative in nature. Sometimes they give unexpected predictions and then they may lead to breakthroughs or deep thinking about assumptions. Sometimes models give prediction completely at variance with observations and then these models have to be drastically revised.

(xi) A model is not good or bad, it does or does not fit: Models may lead to nice and elegant mathematical results, but only those models are acceptable which can explain, predict or control situations. A model may also fit one situation very well and may give a hopeless fit for another situation.

(xii) Modelling forces us to think clearly: Before making a mathematical model, one has to be clear about the structure and essentials of the situation.

(xiii) Sticking to one model may prevent insight: A model helps thinking, but it can also direct thinking in one narrow channel only. Sometimes insight is obtained by breaking with traditional models and designing entirely new ones with new concepts.

(xiv) Inadequate models are also useful: Since they lead us to search for aspects which may have been neglected at first. Failures can be prelude to successes if we can find the reasons for these failures.

(xv) Non-feedback models are improper: A model must include the possibility of its improvement in the light of the experimental or observational data.

(xvi) Partial modelling for subsystems: Before making a model for the whole system, it may be convenient to make partial models for subsystems, test their validity and then integrate these partial models into a complete model. Sometimes existing models are combined to give models for bigger systems. Often models are unified so that the general model includes the earlier models as special cases.

(xvii) Modelling in terms of modules: One may think of models as small modules and by combining them in different ways, one may get models for a large number of systems.

(xviii) Imperfections of models and cost of modelling: No model is perfect and every model can be improved. However each such improvement may cost time and money. The improvement in the model must justify the investment made in this process.

(xix) State variables and relations: For making a mathematical model, one first has to identify the state variables and then specify the relations between them. The right choice of state variables is of the utmost importance.

(xx) Estimation of parameters: Every model contains some parameters and these have to be estimated. The model must itself suggest experiments or observations and the method of calculation of these parameters. Without this explicit specification, the model is incomplete.

(xxi) Validation by independent data: Sometimes parameters are estimated with the help of some data and the same data are used to validate the model. This is illegitimate. Independent data should be used to validate the model.

(xxii) New models to simplify existing complicated models: We start with simple models, introduce more and more variables and functions to make the models more realistic and more complicated and with the additional insights obtained, we should again be able to simplify the complex models.

(xxiii) Modelling $\Rightarrow$ Mathemetics + Discipline: For making a mathematical model of a situation, one must know both mathematics and the discipline in which the situation arises. Efforts to make a mathematical model without deeply understanding the discipline concerned may lead to infructous models. Discipline insight must both precede and follow mathematical modeling.

(xxiv) Transferability of mathematical models: A mathematical model for one field may be equally valid for another field and may be validly transferred to another field, but great care must be exercised in this process. A model which is transferable to a number of fields is very useful, but no model should be thrust on a field unless it is really applicable there.

(xxv) Prediction-validation-iteration cycle: A mathematical model predicts conclusions which are then compared with observations. Usually there is some discrepancy. To remove this discrepancy, we improve the model, again predict and again try to validate and this iteration is repeated till a satisfactory model is obtained.

(xxvi) Models for strategic and tactical thinking: Models may be constructed for determining guidelines for particular situations or for determining an overall strategy applicable to a variety of situations.

(xxvii) Constraints of additivity and normality: Models which are linear, additive and in which the probability distribution follows the normal law are relatively simpler, but relatively more realistic models have to be free from these constraints.

(xxviii) Mathematical modellings and mathematical techniques: Emphasis in applied mathematics has very often been on mathematical techniques, but the heart of applied mathematics is mathematical modeling.

(xxix) Mathematical modeling gives new ideology and unity to applied mathematics: Thus operations, research and fluid dynamics differ in their subject matter as well as in techniques, but mathematical modeling is common to both.

(xxx) Non-uniqueness of models: A situation need not have only one mathematical model and the existence of one model for it should not inhibit the search for better and different models.

(xxxi) Dictionary of mathematical models: It is unlikely that we shall ever have a complete dictionary of mathematical models so that our task will be only to choose an appropriate model for a given situation. Familiarity with existing models will always be useful, but new situations will always demand construction of new models.

(xxxii) No prefabrication of models: Some pure mathematicians believe that every consistent logical structure will one day model some physical situation. This is likely to be an exception rather than the rule. There will always be a very large number of mathematical structures without corresponding physical models and there will always be physical situations without good mathematical models. The search has to go on in both directions. Mathematics for modeling has to be mainly motivated by the world around us.

(xxxiii) Mathematical modeling is an art: It requires experience, insight and understanding. Teaching this art is also another art.

(xxxiv) Criteria for successful models: These include good agreement between predictions and observations of drawing further valid conclusions, simplicity of the model and its precision.

(xxxv) Generality and applicability of models: The Laplace equation model applies to gravitational potential, electro-static potential, irrotational flows and a variety of other situations. There are some models applicable to a wide variety of situations, while there are others which are applicable to specific situations only.

(xxxvi) Unity of disciplines through mathematical modeling: When a number of different situations are represented by the same mathematical model, it reveals a certain identity of structures of these situations. It can lead to a certain economy of efforts and it can reveal a certain underlying unity between different disciplines.

UNIT – II

MATHEMATICAL MODELLING THROUGH DIFFERENTIAL EQUATIONS

INTRODUCTION:

Mathematical modeling through differential equations is a process of translating real-world systems into mathematical language to analyze their behavior. This involves creating a differential equation that represents a system's rates of change, solving it to find a function describing the system's behavior, and then interpreting the solution back in the context of the original problem. This approach is widely used in fields like engineering, physics and biology to understand phenomena from population dynamics to falling objects.

Mathematical modeling in terms of differential equations arises when the situation model involves some *continuous variables* varying with respect to some other continuous variable and we have some reasonable hypotheses about the *rate of change* of dependent variables with respect to independent variables.

When we have one dependent variables x (say population size) depending on one independent variable (say time t), we get a mathematical model in terms of an ordinary *differential equation of the first order*, if the hypothesis is about the rate of change $\frac{dx}{dt}$. The model will be in terms of an *ordinary differential equation of the second order* if the hypothesis involves the rate of change $\frac{dx}{dt}$.

If there are a number of dependent continuous variables and only one independent variables, the hypothesis may give a mathematical model in terms of a *system of first or higher order differential equations.*

If there is one dependent continuous variables and a number of independent continuous variables, we get a mathematical model in terms of a *partial differential equations*. If there are a number of independent continuous variables and a number of independent continuous variables. we can get a mathematical model in terms of *system of partial differential equations.*

LINEAR GROWTH AND DECAY MODELS:

1. Populational Growth Models:

Let $x(t)$ be the population size at time t and let b and d be the birth and death rates. (i.e) the number of individuals born or dying per individual per unit time, then in time interval $(t, t + \Delta t)$, the number of births and deaths would be $bx\Delta t + O(\Delta t)$ and $dx\Delta t + O(\Delta t)$, where $O(\Delta t)$ is an infinitesimal which approaches zero as Δt approaches zero, so that

$$x(t + \Delta t) - x(t) = \big(bx(t) - dx(t)\big)\Delta t + O(\Delta t) \text{ ------------- } (1)$$

so that dividing by Δt and proceeding to the limit as $\Delta t \to 0$, we get

$$\frac{dx}{dt} = (b - d)x = ax \qquad \text{ ------------- } (2)$$

Integrating (2), we get

$$\log x = at + c \qquad \text{-------------- (3)}$$

Using the initial condition $t = 0, x = x_0$

$$\log x_0 = c$$

Substitute the value of c in (3), we get

$$\log x = at + \log x_0$$

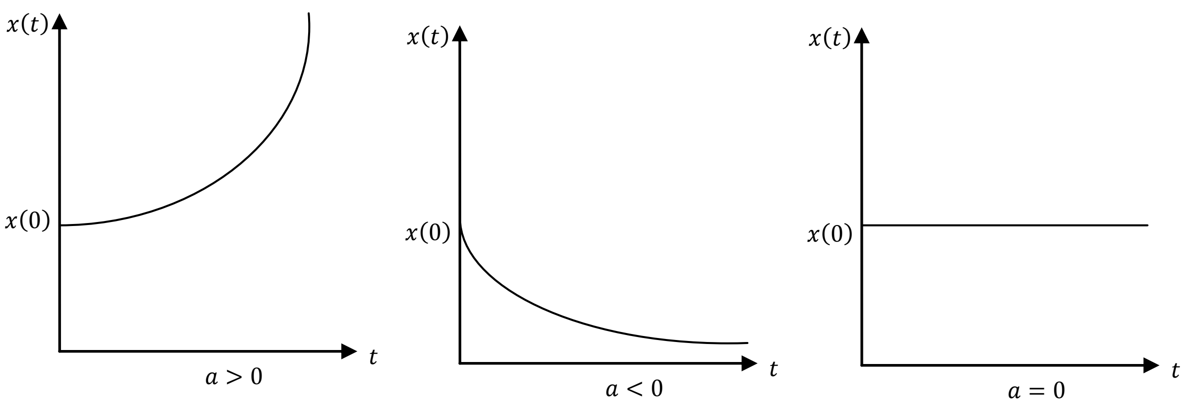
$$\log x - \log x_0 = at$$

$$\log\left(\frac{x}{x_0}\right) = at$$

$$\frac{x}{x_0} = e^{at}$$

$$x = x_0 e^{at} \qquad \text{---------------- (4)}$$

so that the population grows exponentially if $a > 0$, decay exponentially if $a < 0$ and remains constant if $a = 0$



(i) If $a > 0$, the population will become double its present size at time t, where

$$2x(0) = x(0)\exp(aT)$$

$$\exp(aT) = 2$$

Taking log on both sides, we get

$$aT = \log 2$$

$$T = \frac{1}{a}\log 2 = \frac{1}{a}(0.6931) \qquad \text{---------------- (5)}$$

T is called the doubling period of the population and it may be noted that this doubling period is independent of $x(0)$. It depends only on a and is such that the greater the value of a, the smaller the doubling period.

(ii) If $a < 0$, the population will become half its present size in time T'.

when $\frac{1}{2}x(0) = x(0)\exp(aT')$

$$\exp(aT') = \frac{1}{2}$$

Taking log on both sides, we get

$$aT' = \log\frac{1}{2}$$

$$T' = \frac{1}{a}\log\frac{1}{2} \text{ or } \frac{-1}{a}(0.6931) \qquad \text{------------------- (6)}$$

It may be noted that T' is also independent of $x(0)$ and since $a < 0, T' > 0$. T' may be called the half period of the population and it decreases as the excess of death rate over birth rate increases.

GROWTH OF SECIENCE AND SCIENTIST:

Let $S(t)$ denote the number of scientists at time t, $bS(t)\Delta t + O(\Delta t)$ be the number of new scientists trained in time interval $(t, t + \Delta t)$ and let $dS(t)\Delta t + O(\Delta t)$ be the number of scientist who retire from science in the same period, then the previous model applies and the number of scientist should grow exponentially.

The same model applies to the growth of science, mathematics and technology. Thus if $M(t)$ is the amount of mathematics at time t, then the rate of growth of mathematics is proportional to the amount of mathematics, so that

$$\frac{dM}{dt} = aM$$

$$\frac{dM}{M} = adt$$

Integrating, $\log M = at + c$

$$M(t) = e^{at}.e^{c} \qquad \text{------------------ (7)}$$

Using the initial condition $t = 0$, we have

$$M(0) = e^{c}$$

Substitute the value of e^{c} in (7), we get

$$M(t) = M(0)e^{at} \qquad \text{------------------ (8)}$$

Thus according to this model, mathematics, science and technology grow at an exponential rate and double themselves in a certain period of time. During the last two centuries this doubling period has been about ten years. This implies if 1900 we had one unit of mathematics, then in 1910, 1920, 1930, 1940, …, 1980 we have 2, 4, 8, 16, 32, 64, 128, 256 units of mathematics and in 2000 AD we shall have about 1000 units of mathematics. This implies that 99.9% of mathematics that would exist at the end of the present century would have been created in this century and 99.9% of all mathematicians who ever lived would have lived in this century.

The doubling period of mathematics is 10 years and the doubling period of the human population is 30-35 years. These doubling periods cannot obviously be maintained indefinitely because then at some point of time, we shall have more mathematicians then human beings. Ultimately the doubling period of both will be the same, but hopefully this is a long way away.

This model also shows that the doubling period can be shortened by having more intensive training programs for mathematicians and scientists and by creating conditions in which they continue to do creative work for longer durations in life.

EFFECT OF IMMIGRATION AND EMIGRATION ON POPULATION SIZE:

If there is immigration into the population from outside at a rate proportional to the population size, the effect is equivalent to increasing the birth rate. Similarly if there is emigration from the population at a rate proportional to the population size, the effect is the same as that of increase in the death rate.

If however immigration and emigration take place at constant rates i and e respectively, Equ. (2) is modified to

$$\frac{dx}{dt} = bx - dx + i - e$$

$$= ax + k$$

$$\frac{dx}{dt} - ax = k \qquad \text{------------------ (9)}$$

Which is the differential equation of the form $\frac{dy}{dx} + Py = Q$ of the solution is $ye^{\int Pdx} = \int Q\,.\,e^{\int Pdx}dx + c$

Integrating (9), we get

$$xe^{\int -a\,dt} = \int k\,e^{\int -a\,dt}dt + c$$

$$xe^{-at} = \int k\,e^{-at}dt + c$$

$$xe^{-at} = \frac{ke^{-at}}{-a} + c$$

$$x = \frac{k}{-a} + ce^{at} \qquad \text{------------------ (10)}$$

Using the initial condition $t = 0, x = x(0)$

$$x(0) = \frac{k}{-a} + c$$

$$c = x(0) + \frac{k}{a}$$

Substitute the value of c in (10), we get

$$x + \frac{k}{a} = \left(x(0) + \frac{k}{a}\right) e^{at} \qquad \text{------------------- (11)}$$

The model also applies to growth of populations of bacteria and microorganisms, to the increase of volume of timber in a forest, to the growth of malignant cell etc. In the case of forests, planting of new plants will correspond to immigration and cutting of trees will correspond to emigration.

INTEREST COMPOUNDED CONTINUOUSLY:

Let the amount at time t be $x(t)$ and let interest at rate r per unit amount per unit time be compounded continuously and then

$$x(t + \Delta t) = x(t) + rx(t)\Delta t + O(\Delta t)$$

$$x(t + \Delta t) - x(t) = rx(t)\Delta t + O(\Delta t)$$

Dividing Δt and limit $\Delta t \to 0$, we have

$$\lim_{\Delta t \to 0} \frac{\left(x(t+\Delta t) - x(t)\right)}{\Delta t} = \lim_{\Delta t \to 0} \frac{rx(t)\Delta t + O(\Delta t)}{\Delta t}$$

$$\frac{dx}{dt} = rx(t)$$

$$\frac{dx}{dt} - rx = 0$$

Which is the differential equation of the form $\frac{dy}{dx} + Py = Q$ of the solution is $ye^{\int Pdx} = \int Q \,.\, e^{\int Pdx} dx + c.$

$$xe^{\int -rdt} = c$$

$$xe^{-rt} = c$$

$$x = ce^{rt} \qquad \text{------------------ (12)}$$

Using the initial condition $t = 0, x = x(0)$

$$c = x(0)$$

Substitute the value of c in (12), we get

$$x = x(0)e^{rt} \qquad \text{------------------ (13)}$$

This formula can also be derived from the formula for compound interest

$$x(t) = x(0)\left(1 + \frac{r}{n}\right)^{nt} \qquad \text{----------------- (14)}$$

when interest is payable n times per unit time, by taking the limit as $n \to \infty$.

In fact comparison of (13) and (14) gives us two definitions of the transcendental number e viz.

(i) e is the amount of an initial capital of one unit invested for one unit of time when the interest at the unit rate is compounded continuously.

(ii) $e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$ ------------------ (15)

Also from (13) if $x(t) = 1$, then $x(0) = e^{-rt}$ ------------------ (16)

so that e^{-rt} is the present value of a unit amount due one period hence when interest at the rate r per unit amount per unit time is compounded continuously.

RADIOACTIVE DECAY:

Many substances undergo radioactive decay at a rate proportional to the amount of the radioactive substance present at any time and each of them has a half-life period. For uranium 238 it is 4.55 billion years. For potassium it is 1.3 billion years. For thorium it is 13.9 billion years. For rubidium it is 5. billion years while for carbon 14, it is only 5568 years and for white lead it is only 22 years.

In radio geology, these results are used for radioactive dating. Thus the ratio of radio-carbon to ordinary carbon (carbon 12) in dead plants and animals enables us to estimate their time of death. Radioactive dating has also been used to estimate the age of the solar system and of earth as 45 billion years.

DECREASE OF TEMPERATURE:

According to Newton's law of cooling, the rate of change of temperature of a body is proportional to the difference between the temperature T of the body and temperature T, of the surrounding medium, so that

$$\frac{dT}{dt} \propto T - T_s$$

$$\frac{dT}{dt} = k(T - T_s)$$

$$\frac{dT}{dt} = kT - kT_s$$

$$\frac{dT}{dt} - kT = -kT_s$$

Which is the differential equation of the form $\frac{dy}{dx} + Py = Q$ of the solution is $ye^{\int Pdx} = \int Q \, . e^{\int Pdx} dx + c.$

$$Te^{\int -kdt} = \int -k\, T_s e^{\int -kdt} dt + c$$

$$= -kT_s \frac{e^{-kt}}{-k} + c$$

$$Te^{-kt} = T_s e^{-kt} + c$$

$$T = T_s + ce^{-kt} \qquad \text{------------------- (17)}$$

Using the initial condition $t = 0, T = T(0)$

$$T(0) = T_s + c$$

$$c = T(0) - T_s$$

Substitute the value of c in (17), we get

$$T = T_s + (T(0) - T_s)e^{kt} \qquad \text{------------------- (18)}$$

and the excess of the temperature of the body over that of the surrounding medium decays exponentially.

DIFFUSION:

According to Fick's law of diffusion, the time rate of movement of a solute across a thin membrane is proportional to the area of the membrane and to the difference in concentrations of the solute on the two sides of the membrane.

If the area of the membrane is constant and the concentration of solute on one side is kept fixed at a and the concentration of the solution on the other side initially is $c_0 < a$, then Fick's law gives $\frac{dc}{dt} \propto a - c$

$$\frac{dc}{dt} = k(a - c)$$

$$\frac{dc}{dt} + kc = ka$$

Which is the differential equation of the form $\frac{dy}{dx} + Py = Q$ of the solution is $ye^{\int Pdx} = \int Q \, . \, e^{\int Pdx}dx + c.$

$$ce^{\int kdt} = \int k\, ae^{\int kdt}dt + A$$

$$ce^{\int kdt} = \int k\, ae^{kt}dt + A$$

$$= ka\frac{e^{kt}}{k} + A$$

$$ce^{kt} = ae^{kt} + A$$

$$c = a + Ae^{-kt} \qquad \text{------------------ (19)}$$

Using the initial condition $t = 0, c = c(0)$

$c(0) = a + A$

$A = c(0) - a$

Substitute the value of c in (19), we get

$c = a + (c(0) - a)e^{-kt}$

$a - c(t) = \big(a - c(0)\big)e^{-kt}$ ------------------ (20)

and $c(t) \to a$ as $t \to \infty$, whatever be the value of c_0.

CHANGE OF PRICE OF A COMMODITY:

If $p(t)$ is the price of a commodity at time t, then its rate of change is proportional to the difference between the demand $d(t)$ and the supply $s(t)$ of the commodity in the market so that

$\frac{dp}{dt} = k\big(d(t) - s(t)\big)$, where $k > 0$ ------------------- (21)

where $k > 0$, since if demand is more than the supply the price increases. If $d(t)$ and $s(t)$ are assumed linear functions of $p(t)$.

(ie) if $d(t) = d_1 + d_2 p(t), s(t) = s_1 + s_2 p(t), d_2 < 0, s_2 > 0$

$\frac{dp}{dt} = k(d_1 + d_2 p(t) - \big(s_1 + s_2 p(t)\big)$

$= k\big((d_1 - s_1) + (d_2 - s_2)p(t)\big)$

$= k\big(\alpha + \beta p(t)\big)$ where $\alpha = d_1 - s_1$ and $\beta = d_2 - s_2$

$$\frac{dp}{dt} = k\beta\left(\frac{\alpha}{\beta} + p(t)\right)$$

$$\frac{dp}{dt} = K\big(p_e + p(t)\big)$$

$$\frac{dp}{dt} - Kp = Kp_e$$

Which is the differential equation of the form $\frac{dy}{dx} + Py = Q$ of the solution is $ye^{\int Pdx} = \int Q\,.\,e^{\int Pdx}dx + c.$

$$pe^{\int kdt} = \int kp_e e^{\int kdt}dt + c$$

$$pe^{kt} = \int kp_e e^{kt}dt + c$$

$$pe^{kt} = kp_e\frac{e^{kt}}{k} + c$$

$$p = p_e + ce^{-kt} \qquad \text{-------------------- (22)}$$

Using the initial condition $t = 0, p = p(0)$

$$p(0) = p_e + c$$

$$c = p(0) - p_e$$

Substitute the value of c in (22), we get

$$p = p_e + (p(0) - p_e)\, e^{-kt}$$

where p_e is the equilibrium price, so that

$$p_e - p(t) = \left(p_e + p(t)\right) e^{-kt}$$

and $p(t) \to p_e$ as $t \to \infty$.

PROBLEM

1. Suppose the population of the world now is 8 billion and its doubling period is 35 years, what will be the population of the world after 350 years, 700 years, 1050 years? If the surface area of the earth is 1,860,000 billion square feet, how much space would each person get after 1050 years?

Solution:

W.K.T. the population growth $P(t) = P_0 . 2^{t/T_d}$

where P_0 is the initial population, t is the time elapsed and T_d is the doubling period.

Given, the population of the world is 8 billion and doubling period $T_d = 35$ years

After 350 years:

$$P(350) = 8.2^{\frac{350}{35}} = 8(2^{10}) = 8(1024) = 8192 \text{ billion}$$

After 700 years:

$$P(700) = 8.2^{\frac{700}{35}} = 8(2^{20}) = 8(10{,}48{,}576) = 83{,}88{,}608 \text{ billion}$$

After 1050 years:

$$P(1050) = 8.2^{\frac{1050}{35}} = 8(2^{30}) = 8(1{,}07{,}37{,}41{,}824) = 8{,}58{,}99{,}34{,}592 \text{ billion}$$

Now to find the space each person would have, divided the total surface area of the earth by the population after 1050 years. Given, the total surface of the earth is 18,60,000 billion square feet.

$$\text{Space of the person} = \frac{\text{Total surface area}}{\text{population after 1050 years}}$$

$$= \frac{18{,}60{,}000}{8{,}58{,}99{,}34{,}592} \approx 0.0002165 \text{ ft}^2/\text{ person.}$$

2. Find the relation between doubling, tripling and quadrupling times for a population.

Solution:

The relationship can be derived from the exponenetial growth is $P(t) = P_0 e^{kt}$, where $P(t)$ is the population at time t, P_0 is the initial population and k is the growth constant.

Doubling Time (t_d):

$$P(t_d) = 2P_0$$

$$P_0 e^{kt_d} = 2P_0 \implies e^{kt_d} = 2$$

Taking log on both sides, we get

$$kt_d = \log 2$$

$$t_d = \frac{\log 2}{k}$$

Tripling Time (t_t) :

$P(t_t) = 3P_0$

$P_0 e^{kt_t} = 3P_0 \implies e^{kt_t} = 3$

Taking log on both sides, we get

$kt_t = 3$

$t_t = \frac{\log 3}{k}$

Quadrupling Time (t_q) :

$P(t_q) = 4P_0$

$P_0 e^{kt_q} = 4P_0 \implies e^{kt_q} = 4$

Taking log on both sides, we get

$kt_q = 4$

$t_q = \frac{\log 4}{k} = \frac{\log 2^2}{k}$

$t_q = \frac{2 \log 2}{k} = 2t_d$

$\frac{t_q}{t_d} = 2$

3. In an archeological wooden specimen, only 25% of original radio carbon 12 is persent. When was it made? (log $0.25 = -1.386$)

Solution:

Given: 25% of original radio carbon 12 is present.

W.K.T the Radioactive decay is $N(t) = N_0 e^{-\lambda t}$ --------------- (1)

Let half-life period of 14-Carbon, $t_{1/2} = 5730$ years. So $\lambda = \frac{\log 2}{5730}$

From (1) will gives, $\frac{N(t)}{N_0} = e^{-\lambda t}$

$$e^{-\lambda t} = 0.25$$

Taking log on both sides, we get

$$-\lambda t = \log 0.25$$

$$-\lambda t = -1.386 \Longrightarrow t = \frac{1.386}{\lambda}$$

$$t = 1.386 \times \frac{5730}{\log 2}$$

$$\approx 1.386 \times \frac{5730}{0.693} \approx 1.386 \times 8267 = 11{,}465 \text{ years.}$$

4. The rate of change of atmospheric pressure p with respect to height h is assumed proportional to p. If $p = 14.7$psi at $h = 0$ and $p = 7.35$ at $h = 17,500$ feet. What is p at $h = 10,000$ feet?

Solution:

W.K.T the rate of change of atmospheric pressure p w.r.t. height h is $\frac{dp}{dh} = -kp$. The relationship can be derived from the exponenetial growth is $P(h) = P_0 e^{-kh}$.

Given: $p_0 = 14.7$psi at $h = 0$ and $p = 7.35$ at $h = 17{,}500$ feet

To find k:

$$7.35 = 14.7e^{-k(17{,}500)}$$

$$e^{-k(17{,}500)} = \frac{7.35}{14.7}$$

Thaking log on both sides, we get

$$-k(17{,}500) = \log 0.5$$

$$-k(17{,}500) = -0.693 \Longrightarrow k = \frac{0.693}{17{,}500} = 3.96 \times 10^{-5}$$

At $h = 10{,}000$ feet, $p(10{,}000) = 14.7e^{-3.96\times10^{-5}\times10{,}000}$

$= 14.7e^{-0.396}$ since $e^{-0.396} = 0.673$

$= 14.7 \times 0.673 \approx 9.89$psi.

5. What is the rate of interest compounded continuously if a bank's rate of interest is 10% per annum?

Solution:

If rate is 10% per annum compound annually, equivalent continuous rare r satisfies

$$e^r = 1 + 0.10 = 1.10$$

Taking log on both sides, we get

$$r = \log 1.10 \approx 0.0953$$

So, the continuous compounding rate = 9.53% per annum.

6. A body where temperature $\boldsymbol{T}$ is initially $\mathbf{300^0}\boldsymbol{C}$ is placed in a large block of ice. Find its temperature at the end of 2 and 3 minutes.

Solution:

$$T(t) = T_i e^{-kt} + T_0(1 - e^{-kt})$$

Assume surrounding ice $T_0 = 0^0C$, initial $T_i = 300^0C$. So, $T(t) = 300e^{-kt}$

we need k. If not given, assume $k = 0.2\,\text{min}^{-1}$(a typical value for rapid cooling)

For $\boldsymbol{t} = \mathbf{2}$ min, $T(2) = 300e^{-0.2\times 2} = 300e^{-0.4}$

$$= 300 \times 0.6730$$

$$\approx 201.1^0C$$

For $\boldsymbol{t} = \mathbf{3}$ min, $T(3) = 300e^{-0.2\times 3} = 300e^{-0.6}$

$$= 300 \times 0.5488$$

$$\approx 164.6^0C$$

7. The concentration of potassium in a kidney is 0.0025 milligrams per cubic centimeter. The kidney is placed in a large vessel in which the potassium concentration is 0.0040 mg/cm^3. In 1 hour the concentration in the kidney increases to 0.0027 mg/cm^3. After how much time will the concentration be 0.0035 mg/cm^3.

Solution:

Given: Initial concentration $C_0 = 0.0025$ mg/cm^3

Externalconcentration $C_e = 0.0040$ mg/cm^3

After $t = 1$ hours the concentration $C = 0.0027$ mg/cm^3

Let $C(t) = C_e + (C_0 - C_e)e^{-kt}$

$0.0027 = 0.0040 + (0.0025 - 0.0040)e^{-k.1}$

$(0.0027 - 0.0040) = -0.0015e^{-k}$

$-0.0013 = -0.0015e^{-k}$

$e^{-k} = \frac{0.0015}{0.0013} = 0.8667$

$k = -\log 0.8667 = 0.143$

Required time for $C = 0.0035$:

$0.0035 = 0.0040 + (0.0025 - 0.0040)e^{-k.t}$

$-0.0005 = -0.0015e^{-kt}$

$e^{-kt} = \frac{0.0005}{0.0015} = 0.3333$

$kt = -\log 0.3333 = 1.0986$

$t = \frac{1.0986}{0.143} = 7.68$ hours.

8. A population is decaying exponentially. Can this decay be stopped or reversed by immigration at a large constant rate into the population?

Solution:

Let population decay be $\frac{dN}{dt} = -kN + I$, where I is constant immigration rate.

Steady-state occurs when $\frac{dN}{dt} = 0 \Longrightarrow N_{ss} = \frac{I}{k}$.

If the immigration rate is higher than the rate of natural decay, the population will begin to grow instead of decay. It will then approach the new steady state population N_{ss} . Where the immigration and decay are balanced.

NONLINEAR GROWTH AND DECAY MODELS:

1. Logistic law of population growth:

As population increase, due to overcrowding and limitations of resources the birth rate b decreases and the death rate d increase with the population size x. The simplest assumption is to take

$$b = b_1 - b_2x, d = d_1 + d_2x, b_1, b_2, d_1, d_2 > 0$$

From (2), becomes

$$\frac{dx}{dt} = \big((b_1 - b_2x) - (d_1 + d_2x)\big)x$$

$$= \big((b_1 - d_1) - (b_2 + d_2)x\big)x$$

$\frac{dx}{dt} = x(a - bx), a > 0, b > 0$

$\frac{dx}{x(a-bx)} = dt$

Intergrating, $\int \frac{dx}{x(a-bx)} = \int dt$

Consider, $\frac{1}{x(a-bx)} = \frac{A}{x} + \frac{B}{a-bx}$ ----------------- (1)

$\frac{1}{x(a-bx)} = \frac{A(a-bx)+Bx}{x(a-bx)}$

$1 = A(a - bx) + Bx$

Put $x = 0, 1 = Aa \implies A = \frac{1}{a}$

Put $x = \frac{a}{b}, 1 = A\left(a - b\left(\frac{a}{b}\right)\right) + B\left(\frac{a}{b}\right)$

$B = \frac{b}{a}$

Substitute the value of A and B in (1), we get

$\frac{1}{x(a-bx)} = \frac{1}{ax} + \frac{b}{a(a-bx)}$

$\int \frac{dx}{x(a-bx)} = \int \frac{dx}{ax} + \int \frac{b}{a(a-bx)} dx$

$\int dt = \frac{1}{a}\int \frac{dx}{x} + \frac{b}{a}\int \frac{dx}{a-bx}$

$t = \frac{1}{a}\log x + \frac{b}{a}\left(\frac{-\log(a-bx)}{b}\right) - \log c$

$$at = \log x - \log(a - bx) - \log c$$

$$at = \log x - \log c(a - bx)$$

$$at = \log\left(\frac{x}{c(a-bx)}\right)$$

Taking log on both sides, we get

$$\frac{x}{c(a-bx)} = e^{at} \qquad \text{--------------------- (2)}$$

Using initial condition $t = 0$ and $x = x_0$ we have

$$\frac{x_0}{c(a-bx_0)} = 1$$

$$c = \frac{x_0}{a-bx_0}$$

Substitute the c value of (2), we get

$$\frac{x}{\left(\frac{x_0}{a-bx_0}\right)(a-bx)} = e^{at}$$

$$\frac{x(a-bx_0)}{ax_0-bxx_0} = e^{at} \implies ax - bxx_0 = e^{at}(ax_0 - bxx_0)$$

$$ax = e^{at}ax_0 - e^{at}bxx_0 + bxx_0$$

$$e^{at}ax_0 = e^{at}bxx_0 + x(a - bx_0)$$

$$e^{at}ax_0 = x(be^{at}x_0 + a - bx_0)$$

$$x = \frac{e^{at}ax_0}{be^{at}x_0+a-bx_0} = \frac{e^{at}ax_0}{e^{at}(bx_0+(a-bx_0)e^{-at})} = \frac{ax_0}{bx_0+(a-bx_0)e^{-at}}$$

As $t \to \infty$, $x = \frac{ax_0}{bx_0} = \frac{a}{b}$

where $\frac{a}{b}$ is the limiting population (or) carring capacity. At equilibrium state, $\frac{dx}{dt} = 0$

$x(a - bx) = 0$.

Case (i):

When $x = 0$, since linearization about it given $\frac{dx}{dt} \approx x$

(ie) $x = e^{-t}$. so x grows exponenetially from any small initial value.

Therefore, $x = 0$ is unstable.

The other equilibrium $x = \frac{a}{b}$ is table

$$\frac{dx}{dt} = (a - bx)x$$

$$\frac{dx}{dt} = ax - bx^2$$

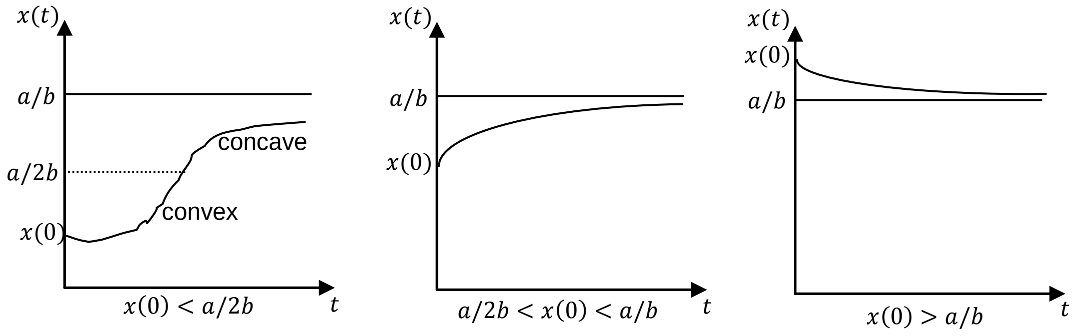
$$\frac{d^2x}{dt^2} = a - 2bx$$

$$a - 2bx = 0 \implies x = \frac{a}{2b}$$

The population of the influence point is exactly half of the carring capacity.

At $x = 0$, $\frac{d^2x}{dt^2} = a > 0$ the population is minimum.

At $x = \frac{a}{b}$, $\frac{d^2x}{dt^2} = a - 2b\left(\frac{a}{b}\right) = -a < 0$ the population is maximum.



- ❖ If $x(0) < \frac{a}{2b}, x(t)$ increases at an increasing rate till $x(t)$ reaches $\frac{a}{2b}$ and then it increases at a decreasing rate and approaches $\frac{a}{b}$ at $t \to \infty$.
- ❖ If $\frac{a}{2b} < x(0) < \frac{a}{b}$, $x(t)$ increases at a decreasing rate and approaches $\frac{a}{b}$ at $t \to \infty$.
- ❖ If $x(0) = \frac{a}{b}$, $x(t)$ is always equal to $\frac{a}{b}$.
- ❖ If $x(0) > \frac{a}{b}$, $x(t)$ decreases at a decreasing absolute rate and approaches $\frac{a}{b}$ as $t \to \infty$.

SPREAD OF TECHNOLOGICAL INNOVATIONS AND INFECTIOUS DISEASES:

Let $N(t)$ be the number of companies which have adopted a technological innovation till time t, then the rate of change of the number of these companies depends both on the number of companies which have adopted this innovation and on the number of those which have not yet adopted it, so that if R is the total number of companies in the region

$$\frac{dN}{dt} = kN(R - N)$$

$$\int \frac{dN}{N(R-N)} = \int k dt \qquad \text{------------------ (1)}$$

Take, $\frac{1}{N(R-N)} = \frac{A}{N} + \frac{B}{R-N}$ ------------------ (2)

$$1 = A(R-N) + BN$$

Put $N = 0$, we get

$$A = \frac{1}{R}$$

Put $N = R$, we get

$$B = \frac{1}{R}$$

From (2), $\frac{1}{N(R-N)} = \frac{1}{NR} + \frac{1}{R(R-N)}$

By Integrating, $\int \frac{dN}{N(R-N)} = \int \frac{dN}{NR} + \int \frac{dN}{R(R-N)}$

$$= \frac{1}{R}\log N - \frac{1}{R}\log(R-N)$$

Substitute the value of (1), we get

$$\frac{1}{R}\log N - \frac{1}{R}\log(R-N) = \int kdt$$

$$\frac{1}{R}\log\left(\frac{N}{R-N}\right) = kt + c$$

$$\log\left(\frac{N}{R-N}\right) = Rkt + Rc$$

Taking log on both sides, we get

$$\frac{N}{R-N} = e^{Rkt+Rc}$$

$$\frac{N}{R-N} = e^{Rkt}.e^{Rc} = C.e^{Rkt} \quad \text{-----------------} \ (3)$$

Using the initial condition $t = 0, N = N_0$

$$C = \frac{N_0}{R-N_0}$$

Substitute the value of C in (3), we get

$$\frac{N}{R-N} = \left(\frac{N_0}{R-N_0}\right) e^{Rkt}$$

$$N(R - N_0) = N_0(R - N)e^{Rkt}$$

$$NR - NN_0 = RN_0e^{Rkt} - NN_0e^{Rkt}$$

$$NR - NN_0 + NN_0e^{Rkt} = RN_0e^{Rkt}$$

$$N = \frac{RN_0e^{Rkt}}{R-N_0+N_0e^{Rkt}} = \frac{RN_0e^{Rkt}}{e^{Rkt}(Re^{-Rkt}-N_0e^{-Rkt}+N_0)}$$

Similarly if $N(t)$ is the umber of infected persons, the rate at which the number of infexted persons increases depends on the product of the numbers of infested and susceptible, where R is the total number of persons in the system. While $N(t)$ is essentially an integer-valued varible, we have treated as a continuous variblaes. This can be regard as an idealization of the situation or as an approximation to reality.

RATE OF DISSOLUTION:

Let $x(t)$ be the amount of undissolved solute in a solvent at time t and let c_0 be the maximum concentration or saturation concentration, (ie) the maximum amount of the solute that can be dissolved in a unit volume of the solvent. Let V be the volume of the solvent. It is found that the rate at which the solute is dissolved ismproportional to the amount of undissolved solute and to the difference between the concentration of the solute at time t and the maximum possible concentration, so that we get

$$\frac{dx}{dt} = kx(t)\left(\frac{x(0)-x(t)}{V} - c_0\right)$$

$$= kx(t)\left(\frac{x(0)-x(t)-Vc_0}{V}\right)$$

$$= \frac{kx(t)}{V}\left((x_0 - c_0V) - x(t)\right)$$

LAW OF MASS ACTION: CHEMICAL REACTIONS

Two chemical substances combine in the ratio $a{:}b$ to form a thirs substance Z. If $z(t)$ is the amount of the third substance at time t, then a proportion $\frac{az(t)}{a+b}$ of it consists of the first substance and a proportion $\frac{bz(t)}{a+b}$ of it consists of the second substance. The rate of formation of the third substance is proportional to the product of the amount of the two component substances which have not yet combined together. If A and B are the initial amounts of the two substances, then we get

$$\frac{dz}{dt} = k\left(A - \frac{az}{a+b}\right)\left(B - \frac{bz}{a+b}\right)$$

$$\frac{dz}{dt} = \frac{kab}{(a+b)^2}\left(\frac{(a+b)A}{a} - z\right)\left(\frac{(a+b)B}{b} - z\right)$$

$$\frac{dz}{dt} = k_1(A_1 - z)(B_1 - z)$$

$$\frac{dz}{(A_1-z)(B_1-z)} = k_1 dt$$

$$\int \frac{dz}{(A_1-z)(B_1-z)} = \int k_1 dt \qquad \text{----------------- (1)}$$

Take, $\frac{1}{(A_1-z)(B_1-z)} = \frac{x}{A_1-z} + \frac{y}{B_1-z}$

$$1 = x(B_1 - z) + y(A_1 - z)$$

Put $z = A_1$, we get

$$x = \frac{1}{B_1-A_1}$$

Put $z = B_1$, we get

$$y = \frac{1}{A_1-B_1}$$

$$\frac{1}{(A_1-z)(B_1-z)} = \frac{1}{(B_1-A_1)(A_1-z)} + \frac{1}{(A_1-B_1)(B_1-z)}$$

Taking integral on both sides, we get

$$\int \frac{dz}{(A_1-z)(B_1-z)} = \int \frac{dz}{(B_1-A_1)(A_1-z)} + \int \frac{dz}{(A_1-B_1)(B_1-z)}$$

$$= \frac{1}{A_1-B_1}(\log(A_1 - z) - \log(B_1 - z))$$

$$= \frac{1}{A_1-B_1} \log \frac{A_1-z}{B_1-z} \qquad \text{---------------- (2)}$$

Equating the (1) and (2), we get

$$\frac{1}{A_1-B_1}\log\frac{A_1-z}{B_1-z} = \int k_1 dt$$

$$\frac{1}{A_1-B_1}\log\frac{A_1-z}{B_1-z} = k_1 t + c$$

$$\log\frac{A_1-z}{B_1-z} = (A_1 - B_1)(k_1 t + c)$$

$$\log\frac{A_1-z}{B_1-z} = (A_1 - B_1)k_1 t + c(A_1 - B_1)$$

$$\log\frac{A_1-z}{B_1-z} = (A_1 - B_1)k_1 t + C$$

$$\frac{A_1-z}{B_1-z} = e^{(A_1-B_1)k_1 t + C}$$

$$A_1 - z = (B_1 - z)e^{(A_1-B_1)k_1 t}.C$$

$$A_1 - CB_1 e^{(A_1-B_1)k_1 t} = z\left(1 - Ce^{(A_1-B_1)k_1 t}\right)$$

$$z = \frac{A_1 - CB_1 e^{(A_1-B_1)k_1 t}}{1 - Ce^{(A_1-B_1)k_1 t}}$$

This is the nonlinear differential equation for a second order reaction. Similarly for an n^{th} order reaction, we get the nonlinear equation

$$\frac{dz}{dt} = k(A_1 - a_1 z)(A_2 - a_2 z) \ldots (A_n - a_n z)$$

where $a_1 + a_2 + \cdots + a_n = 1$.

PROBLEM

1. Substances X and Y combine in the ratio $2:3$ to from Z.When 45grams of X and 60grams of Y are mixed together, 50grams of Z are formed in 5 minites. How many grams of Z will be found in 210 minutes? How much time will it like to get 70grams of Z?

Solution:

The problem states that 50 grams of substance Z are formed in 5 minutes. This gives us a rate of reaction is

$$\text{Rate} = \frac{50}{5} = 10\text{g/min}$$

The ratio of substances X and Y combining to form Z is $2:3$. This means that for every 2 parts of X and 3 parts of Y, 5 parts of Z are formed (by mass). This information about the limiting reactant is not needed to answer the specific questions posed, as the rate of formation is already given. We can determine the mass of Z produced in 210 minutes.

$$\text{Mass of Z} = \text{Rate} \times \text{Time}$$

$$\text{Mass of Z} = 10 \times 210$$

$$\text{Mass of Z} = 2100\text{g}$$

Now, we can use the same rate of formation to find the time required to produce a specific amount of Z. To calculate the time to form 70grams of Z. Then,

$$\text{Time} = \frac{\text{Mass of Z}}{\text{Rate}} = \frac{70}{10} = 7\text{min}.$$

2. Cigarette consumption in a country increase from 50 per capita in 1900 AD to 3900 per capita in 1960 AD. Assuming that the growth in consumption follows a logistic law with a limiting consumption of 4000 per capita, estimate the consumption per capita in 1950.

Solution:

Given: $K = 4000$

W.K.T. the logistic growth model is given by the equation $N(t) = \frac{K}{1+Ce^{-rt}}$ ---- (1)

Where $N(t)$ is the population at time t, K is the limiting consumption and C and r are constant.

From (1), $N(t) = \frac{4000}{1+Ce^{-rt}}$ ------------------- (2)

Given the consumption in 1900 at $t = 0$ then, $N(0) = 50$

From (2), $50 = \frac{4000}{1+C}$

$$1 + C = \frac{4000}{50} = 80$$

$$c = 79$$

Now, we use the consumption value for 1960 at $t = 60$ then, $N(60) = 3900$

From (2), $3900 = \frac{4000}{1+79e^{-r(60)}}$

$$1 + 79e^{-60r} = \frac{4000}{3900} = \frac{40}{39}$$

$$79e^{-60r} = \frac{40}{39} - 1 = \frac{1}{39}$$

$$e^{-60r} = \frac{1}{39 \times 79} = \frac{1}{3081}$$

Taking log on both sides, we get

$$-60r = -\log 3081$$

$$60r = 8.0336$$

$$r = \frac{8.0336}{60} \approx 0.1339$$

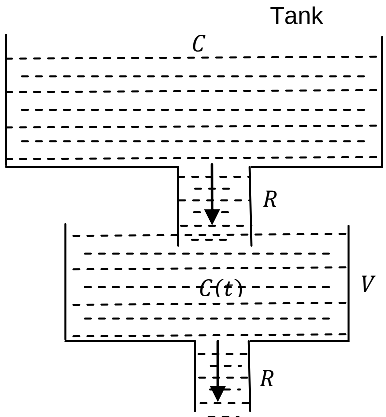
To find the consumption in 1950, (ie) $P(50)$:

From (2), $N(50) = \frac{4000}{1+79e^{-0.1339 \times 50}}$

$$= \frac{4000}{1+79(0.000788)} = \frac{4000}{1+0.06227} \approx 3765.6 \approx 3766$$

COMPARTMENT MODELS:

In the last two sections, we got mathematical models in terms of ordinary differential equations of the first order, in which all variables were seperable. In the present section, we got models in terms of linear differential equations of the first order.



A SIMPLE COMPARTMENT MODEL:

Let a vessel contain a volume V of a solution with concentration $c(t)$ of a substance at time t. Let a solution with constant concentration C in an overhead tank enter the vessel at a constant rate R and after mixing thoroughly with the solution in the vessel, let the mixture with concentration $c(t)$ leave at the same rate R so that the volume of the solution in the vessel remains V.

Using the principle of continuity, we get

$$V\big(c(t+\Delta t)-c(t)\big)=RC\Delta t-Rc(t)\Delta t+O(\Delta t)$$

giving, $V\frac{dc}{dt}+Rc=RC$

Dividing V, $\frac{dc}{dt}+\frac{R}{V}c=\frac{RC}{V}$

Which is the differential equation of the form $\frac{dy}{dx}+Py=Q$ of the solution is $ye^{\int Pdx}=\int Q\,.e^{\int Pdx}dx+c.$

$$ce^{\int\frac{R}{v}dt}=\int\frac{R}{V}Ce^{\int\frac{R}{V}dt}dt+A$$

$$ce^{\frac{R}{V}t}=\frac{RC}{V}\int e^{\frac{R}{V}t}dt+A$$

$$ce^{\frac{R}{V}t}=\frac{RC}{V}\left(\frac{e^{\frac{R}{V}t}}{R/V}\right)+A$$

$$ce^{\frac{R}{V}t}=Ce^{\frac{R}{V}t}+A$$

$$c=C+Ae^{-\frac{R}{V}t} \qquad \text{------------------- (1)}$$

Using the initial conditions $t=0$ and $c=c(0)$, we get

$$c(0)=C+A$$

$$A=c(0)-C$$

Substitute the value of A in (1), we get

$$c(t) = C + (c(0) - C)e^{-\frac{R}{V}t}$$

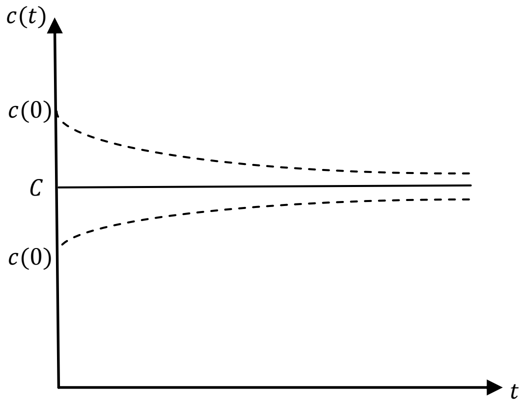
$$c(t) = C - \big(C - c(0)\big)e^{-\frac{R}{V}t} \qquad \text{------------------- (2)}$$

If $C > c_0$, the concentration in the vessel increases to C, on the onther hand if $C < c_0$ the concentration in the vessel decreases to C.

If the rate R' at which the solution leaves the vessel is less than R, the equation of continuity gives

$$\frac{d}{dt}\big((V_0 + (R - R')t)c(t)\big) = RC - R'c(t)$$

where V is the initial volume of the solution in the vessel. This is also a linear differential equation of the first order.



DIFFUSION OF GLUCOSE OR A MEDICINE IN THE BLOODSTREAM:

Let the volume of blood in the human body be V and let the initial concentration of glucose in the bloodstream be $c(0)$. Let glucose be introduced in the bloodstream at a constant rate I. Glucose is also removed from the bloodstream due to the physiological needs of the human body at a rate proportional to $c(t)$, so that the continuity principle gives

$$V\frac{dc}{dt} = I - kc$$

Now let a dose D of a medicine be given to a patient at regular intervals of duration T each. If the medicine also disappears from the system at a rate proportional to $c(t)$, the concentration of the medicine in the bloodstream, then the differential equation given by the continuity principle is

$$V\frac{dc}{dt} = -kc$$

$$\frac{dc}{c} = -\frac{k}{V}dt$$

Integrating, $\int \frac{dc}{c} = \int -\frac{k}{V}dt$

$$\log c = -\frac{k}{V}t + A$$

$$c = e^{-\frac{k}{V}t+A}$$

$$c = e^{-\frac{k}{V}t}.e^{A}$$

$$c = Be^{-\frac{k}{V}t} \qquad \text{---------------------- (1)}$$

using the initial condition $t = 0$ and $c(0) = D$

$$c(0) = B \implies D = B$$

From (1), $c = De^{-\frac{k}{V}t}$, $0 \leq t \leq T$ ---------------------- (2)

At time T, the residue of the first dose is $De^{-\frac{k}{V}t}$ and now another dose D is given so that $c = \left(De^{-\frac{k}{V}T} + D\right)e^{-\frac{k}{V}(t-T)}$

$$= De^{-\frac{k}{V}t} + De^{-\frac{k}{V}(t-T)}, T \leq t < 2T$$

The first term gives the residual of the first dose and the second term gives the residual of the second dose. Proceesing in the same way, we get after n doses have been given

$$c(t) = De^{-\frac{k}{V}t} + De^{-\frac{k}{V}(t-T)} + De^{-\frac{k}{V}(t-2T)} + \cdots + De^{-\frac{k}{V}(t-(n-1)T)}$$

$$= De^{-\frac{k}{V}t}\left(1 + e^{\frac{kT}{V}} + e^{\frac{2kT}{V}} + \cdots + e^{\frac{(n-1)kT}{V}}\right)$$

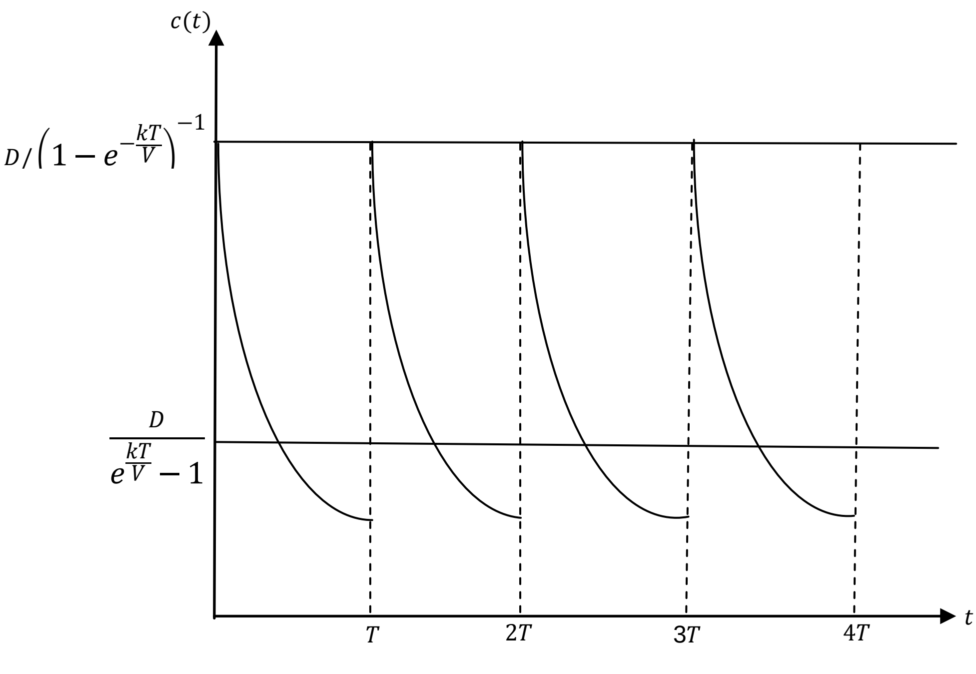
$$= De^{-\frac{k}{V}t}\left(1 + e^{\frac{kT}{V}} + \left(e^{\frac{kT}{V}}\right)^2 + \cdots + \left(e^{\frac{kT}{V}}\right)^{n-1}\right)$$

$$= De^{-\frac{k}{V}t}\left(\frac{e^{\frac{nkT}{V}}-1}{e^{\frac{kT}{V}}-1}\right), (n-1)T \le t < nT$$

$$c(nT-0) = D\,\frac{1-e^{-\frac{nkT}{V}}}{e^{\frac{kT}{V}}-1}$$

$$c(nT+0) = D\,\frac{e^{\frac{kT}{V}}-e^{-\frac{nkT}{V}}}{e^{\frac{kT}{V}}-1}$$

Thus the concentration never exceeds $\frac{D}{1-e^{-\frac{kT}{V}}}$. The graph of $c(t)$ is shown in below



Thus in each interval, concentration decreases. In any interval, the concentration is maximum at the beginnig of this interval and thus maximu concentration at the beginnig of an interval goes on increasing as the number of intervals increases, but the maximum value is always below $\frac{D}{1-e^{-\frac{kT}{V}}}$. The minimum value in an interval occurs at the end of each interval. This also increases, but it lies below $\frac{D}{e^{\frac{kT}{V}}-1}$.

The concentration curve is piecewise continuous and has points of discontinuity at $T, 2T, 3T, \ldots$.

By injecting glucose or penicillin into blood and fitting curve (2) to the data, we can estimate the value of k and V. In particular this gives a mothod for finding the volume of blood in the human body.

THE CASE OF A SUCCESSION OF COMPARTMENTS:

Let a solution with concentration $c(t)$ of a solute pass successively into n tanks in which the initial concentrations of the solution are $c_1(0), c_2(0), \ldots, c_n(0)$. The rate of inflow in each tank is the same as the rate of outflow from the tank. We have to find the concentrations $c_1(t), c_2(t), \ldots, c_n(t)$ at time t. We get the equations

$$V\frac{dc_1}{dt} = Rc - Rc_1$$

$$V\frac{dc_2}{dt} = Rc_1 - Rc_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$V\frac{dc_n}{dt} = Rc_{n-1} - Rc_n$$

By solving the first of these equations, we get $c_1(t)$. Substituting the value of $c_1(t)$ and proceeding in the same way, we can find $c_3(t), \ldots, c_n(t)$.

PROBLEM

1. Let $G(t)$ be the amount of glucose present in the bloodstream of a patient at time t. Assuming that the glucose is injected into the bloodstream at a constant rate of C grams per minute, and at the same time is converted and removed from the blood stream at a rate proportional to the amount of glucose present, find the amount $G(t)$ at any time t. If $G(0) = G_0$, what is the equilibrium level of glucose in the bloodstream?

Solution:

The rate of change of glucose is given by $\frac{dG}{dt} = C - kG$

$$\frac{dG}{C-kG} = dt$$

Integrating, $\int \frac{dG}{C-kG} = \int dt$

$$-\frac{1}{k}\log(C - kG) = t + A$$

$$\log(C - kG) = -kt - kA$$

Taking log on both sides, we get

$$C - kG = e^{-kt-kA}$$

$$C - kG = e^{-kt}.e^{-kA}$$

$$C - kG = A_1 . e^{-kt}$$

$$kG = C - A_1 . e^{-kt}$$

$$G = \frac{C}{k} - \frac{A_1}{k} e^{-kt} = \frac{C}{k} + Be^{-kt}$$

$$G(t) = \frac{C}{k} + Be^{-kt} \quad \text{where } B = \frac{A_1}{k} \qquad \text{------------------ (1)}$$

Using the initial condition $t = 0$ and $G(t) = G_0$

$$G(0) = \frac{C}{k} + B$$

$$B = G_0 - \frac{C}{k}$$

Substitute the B value in (1), we get

$$G(t) = \frac{C}{k} + \left(G_0 - \frac{C}{k}\right) e^{-kt}$$

2. A patient was given 5 micro-Curies (μci) of a type of iodine. Two hours later 0.5μci had been taken up by his thyroid. How much would bave been taken by the thyroid in two hours if he had been given 15μci?

Solution:

Given the initial dose is 5μci and two hours later 0.5μci amount of his tyroid.

$$\text{Rate} = \frac{\text{Absorbed Amount}}{\text{Total type of idodine}} = \frac{0.5}{5} = 0.1\mu\text{ci}$$

The new amount of iodine given by 15μci.

New amount of dose $= 0.1 \times 15 = 1.5\mu\text{ci}$

3. A gene has two alleles $\boldsymbol{A}$ and $\boldsymbol{a}$ which occur in proportional $\boldsymbol{p(t)}$ and $\boldsymbol{q(t) = 1 - p(t)}$ respectively in the population at time $\boldsymbol{t}$. Suppose that allele $\boldsymbol{A}$ mutates to $\boldsymbol{a}$ at a constant rate $\boldsymbol{m}$. If $\boldsymbol{p(0) = q(0) = \frac{1}{2}}$, find $\boldsymbol{p(t)}$ and $\boldsymbol{q(t)}$. Write the equations when both alleles can mutate into each other at different rates.

Solution:

The rate of change of alleles frequency $p(t)$ is $\frac{dp}{dt} = -mp(t)$

$$\frac{dp}{p} = -mdt$$

Integrating, $\log p = -mt + c$

Taking log on both sides, we have

$$p = e^{-mt+c} = e^{-mt}.e^{c}$$

$$p(t) = Ce^{-mt} \qquad \text{------------- (1)}$$

Using initial contions $t = 0$ and $p(0) = \frac{1}{2}$, we have

From (1), $p(0) = C \Longrightarrow C = \frac{1}{2}$

Substitute the value of C in (1), we get

$$p(t) = \frac{1}{2}e^{-mt}$$

Since $q(t) = 1 - p(t)$, then $q(t) = 1 - \frac{1}{2}e^{-mt}$ -------------- (2)

Let the rate of change of $p(t)$ is now a balance between the loss of A alleles (due to mutation to a) and the gain of A alleles (due to back-mutation from a)

$$\frac{dp}{dt} = \mu_1 q(t) - \mu_2 p(t)$$

Since $q(t) = 1 - p(t)$, then $\frac{dp}{dt} = \mu_1\big(1 - p(t)\big) - \mu_2 p(t)$

$$\frac{dp}{dt} = \mu_1 - \mu_1 p(t) - \mu_2 p(t)$$

$$= \mu_1 - (\mu_1 + \mu_2)p(t)$$

Simillarly, $\frac{dq}{dt} = \mu_2 p(t) - \mu_1 q(t)$

Since $q(t) = 1 - p(t)$, then $\frac{dq}{dt} = \mu_2 p(t) - \mu_1\big(1 - p(t)\big)$

$$\frac{dq}{dt} = \mu_2 p(t) - \mu_1 + -\mu_1 p(t)$$

$$= -\mu_1 + (\mu_2 - \mu_1)p(t)$$

$$= -\big(\mu_1 - (\mu_1 + \mu_2)p(t)\big)$$

$$= -\frac{dp}{dt}$$

Therefore, the equations when both alleles can mutate into each other at different rates.

4. A lake of constant volume V contains at time t an amount $Q(t)$ of pollutant evenly distributed throughout the lake. Suppose water containing concentration k of pollutant enters the lake at a rate r and water leaves the lake at the same rate. Suppose pollutants are also added to the lake at a constant rate P. If initial concentration of the pollutant in the lake is c_0, find $c(t)$.

Solution:

The rate of change of $Q(t)$ is $\frac{dQ}{dt} = (rk + P) - \frac{rQ}{V}$ --------------- (1)

Let the constant volume of lake is V and the concentration of the pollutant is $c(t) = \frac{Q(t)}{V} \Longrightarrow Q(t) = Vc(t)$ --------------- (2)

Substitute (2) in (1), we get

$$V\frac{dc}{dt} = (rk + P) - \frac{rcV}{V}$$

$$\frac{dc}{dt} = \frac{rk+P}{V} - \frac{rc}{V}$$

$$\frac{dc}{dt} = \frac{r}{V}\left(k + \frac{P}{r}\right) - \frac{rc}{V}$$

$$\frac{dc}{dt} + \frac{r}{V}c = \frac{r}{V}\left(k + \frac{P}{r}\right) \quad \text{---------------- (3)}$$

This equation is of the form $\frac{dc}{dt} + p(t)c = q(t)$ is the solution $c.(\mathrm{I.F}) = Q.(\mathrm{I.F}) + C$

Integratinf Factor, $\mathrm{I.F} = e^{\int p(t)dt}$

$$\mathrm{I.F} = e^{\int \frac{r}{V}dt} = e^{\frac{r}{V}t}$$

Multiply by $e^{\frac{r}{V}t}$ in (3), we get

$$e^{\frac{r}{V}t}\frac{dc}{dt} + e^{\frac{r}{V}t}\frac{r}{V}c = e^{\frac{r}{V}t}\frac{r}{V}\left(k + \frac{P}{r}\right)$$

$$\frac{d}{dt}\left(c.e^{\frac{r}{V}t}\right) = e^{\frac{r}{V}t}\frac{r}{V}\left(k + \frac{P}{r}\right)$$

Integrating, $\int \frac{d}{dt}\left(c.e^{\frac{r}{V}t}\right) = \int \frac{r}{V}\left(k + \frac{P}{r}\right)e^{\frac{r}{V}t}dt$

$$c.e^{\frac{r}{V}t} = \frac{r}{V}\left(k + \frac{P}{r}\right)\left(\frac{e^{\frac{r}{V}t}}{r/V}\right) + C$$

$$c.e^{\frac{r}{V}t} = \left(k + \frac{P}{r}\right)\left(e^{\frac{r}{V}t}\right) + C$$

Dividing $e^{\frac{r}{V}t}$ the above equation, we get

$$c = \left(k + \frac{P}{r}\right) + Ce^{-\frac{r}{V}t}$$

UNIT – III

MATHEMATICAL MODELLING THROUGH SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

MATHEMATICAL MODELING IN POPULATION DYNAMICS

1. Prey-Predator Models:

Let $x(t), y(t)$ be the populations of the prey and predator species at time t. We assume that

(i) if there are no predators, the prey species will grow at a rate proportional to the population of the prey species.

(ii) if there are no prey, the predator species will decline at a rate proportional to the population of the predator species.

(iii) the presence of both predator and presy is beneficial to the growth of the predator species and is harmful to the growth of the prey species. More specifically the predator species and is harmful to the growth of the prey species. More specifically the predator species increases and the prey species decreases at rates proportional to the product of the two populations.

These assumptions give the systems of nonlinear first order ordinary differential equations

$$\frac{dx}{dt} = ax - bxy$$

$$= x(a - by), a, b > 0 \qquad \text{---------------- (1)}$$

$$\frac{dy}{dt} = -py + qxy$$

$$= -y(p - qx), p, q > 0 \qquad \text{---------------- (2)}$$

Both $\frac{dx}{dt}$ and $\frac{dy}{dt}$ is vanish if (1) and (2), we get

$x(a - by) = 0$ $\qquad\qquad$ $-y(p - qx) = 0$

$x = 0$ $\qquad\qquad$ $y = 0$

$y = \frac{a}{b}$ $\qquad\qquad$ $x = \frac{p}{q}$

If the initial populations of prey and predator species are $\frac{p}{q}$ and $\frac{a}{b}$ respectively, the populations will not change with time. These are the equilibrium sizez of the populations of the two species. Of course $x = 0, y = 0$ also gives another equilibrium position.

From (1) and (2), $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{y(p-qx)}{x(a-by)}$

$$\frac{a-by}{y} dy = -\frac{p-qx}{x} dx, \quad x_0 = x(0), y_0 = y(0)$$

Taking Intagrating on both sides, we get

$$\int_{y_0}^{y} \frac{a-by}{y} dy = \int_{x_0}^{x} -\frac{p-qx}{x} dx$$

$$\int_{y_0}^{y} \left(\frac{a}{y} - b\right) dy = -\int_{x_0}^{x} \left(\frac{p}{x} - q\right) dx$$

$$a(\log y)_{y_0}^{y} - (by)_{y_0}^{y} = -p(\log x)_{x_0}^{x} + (qx)_{x_0}^{x}$$

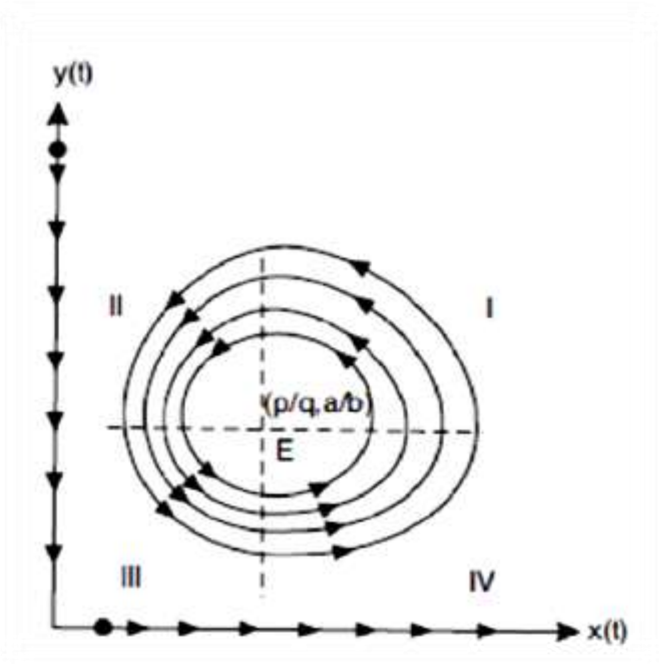
$$a(\log y - \log y_0) - b(y - y_0) = -p(\log x - \log x_0) + q(x - x_0)$$

$$a\log\left(\frac{y}{y_0}\right) - b(y - y_0) = -p\log\left(\frac{x}{x_0}\right) + q(x - x_0)$$

$$a\log\left(\frac{y}{y_0}\right) + p\log\left(\frac{x}{x_0}\right) = b(y - y_0) + q(x - x_0)$$

Thus through every point of the first quadrant of the xy-plane, there is a unique trajectory. No two trajectories can intersect, since intersection will imply two different slopes at the same point.

If we start with (0,0) or $\left(\frac{p}{q}, \frac{a}{b}\right)$, we get point trajectories. If we start with $x = x_0$, $y = y_0$ from Eqns. (1) and (2), we find that x increases while y remains zero. Similarly, if we start with $x = 0, y = y_0$, we find that x remains zero while y decreases. Thus positive axes of x and y give two line trajectories.



Since no two trajectories intersect, no trajectory starting from a point situated within the first quardrant will intersect the x axis and y axis trajectories. Thus all trajectories corresponding to positive initial populations will lie strictly within the first quadrant. Thus if the initial populations are positive, the populations will be always positive. If the population of one (or both) species is initially zero, it will always ramain zero.

The lines through $\left(\frac{p}{q}, \frac{a}{b}\right)$ parallel to the axes of coordinates divide the first quadrant into four parts I, II, III and IV. Using Eqns. (1), (2), we find that

In I	$\frac{dx}{dt} < 0$	$\frac{dy}{dt} > 0$	$\frac{dy}{dx} < 0$
In II	$\frac{dx}{dt} < 0$	$\frac{dy}{dt} < 0$	$\frac{dy}{dx} > 0$
In III	$\frac{dx}{dt} > 0$	$\frac{dy}{dt} < 0$	$\frac{dy}{dx} > 0$
In IV	$\frac{dx}{dt} > 0$	$\frac{dy}{dt} > 0$	$\frac{dy}{dx} > 0$

This gives the direction field at all points as shown in above Figure. Each trajectory is a closed convex curve. These trajectories appear relatively cramped near the axes.

In I and II, the prey species decreases and in III and IV, it increases. Similarly in IV and I, the predator species increases and in II and III, it decreases. After a certain period, both species return to their original sizes and thus both species' sizes vary periodically with time.

2. COMPETITION MODELS:

Let $x(t)$ and $y(t)$ be the populations of two species competing for the same resouces, then each species grows in the absence of the other species, and the rate of growth of each species decreases due to the presence of the other species'. This gives the system of differential equations

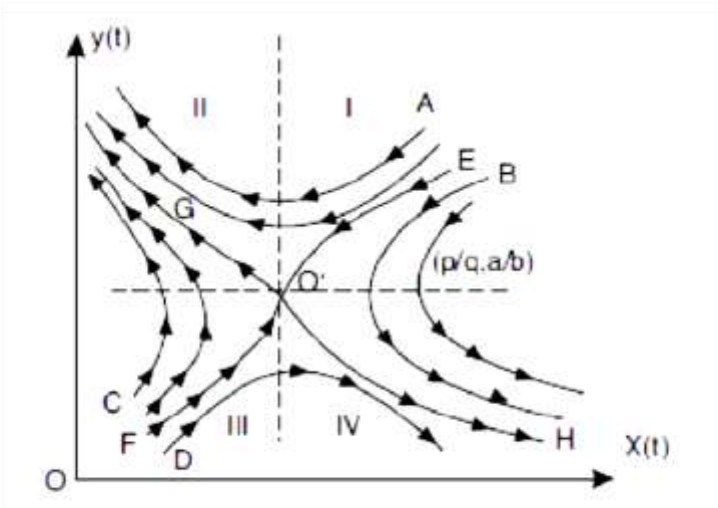
$$\frac{dx}{dt} = ax - bxy = bx\left(\frac{a}{b} - y\right);\ a > 0, b > 0 \text{ ------------------- (3)}$$

$$\frac{dy}{dt} = py - qxy = qy\left(\frac{p}{q} - x\right); p > 0, q > 0 \text{ -------------------- (4)}$$

If $\frac{dx}{dt} = 0$, then $x = 0$ and $y = \frac{a}{b}$

If $\frac{dy}{dt} = 0$, then $y = 0$ and $x = \frac{p}{q}$

Therefore, these are equilibrium positions viz $(0, 0)$ and $\left(\frac{p}{q}, \frac{a}{b}\right)$. There are two point trajectories viz. $(0, 0)$ and $\left(\frac{p}{q}, \frac{a}{b}\right)$ and there are two line trajectories viz. $x = 0$ and $y = 0$.



In I	$\frac{dx}{dt} < 0$	$\frac{dy}{dt} > 0$	$\frac{dy}{dx} < 0$
In II	$\frac{dx}{dt} < 0$	$\frac{dy}{dt} < 0$	$\frac{dy}{dx} > 0$
In III	$\frac{dx}{dt} > 0$	$\frac{dy}{dt} < 0$	$\frac{dy}{dx} > 0$
In IV	$\frac{dx}{dt} > 0$	$\frac{dy}{dt} > 0$	$\frac{dy}{dx} > 0$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y(p-qx)}{x(a-by)}$$

$$\frac{a-by}{y}dy = \frac{p-qx}{x}dx$$

Integrating, $\int_{y_0}^{y} \frac{a-by}{y}dy = \int_{x_0}^{x} \frac{p-qx}{x}dx$

$$\int_{y_0}^{y} \left(\frac{a}{y} - b\right) dy = \int_{x_0}^{x} \left(\frac{p}{x} - q\right) dx$$

$$(a\log y - by)_{y_0}^{y} = (p\log x - qx)_{x_0}^{x}$$

$$a\log y - by - a\log y_0 + by_0 = p\log x - qx - p\log x_0 + qx_0$$

$$a\log\left(\frac{y}{y_0}\right) - b(y - y_0) = p\log\left(\frac{x}{x_0}\right) - q(x - x_0)$$

the trajectory which passes through $\left(\frac{p}{q}, \frac{a}{b}\right)$ is

$$a\log\left(\frac{by}{a}\right) - by + a = p\log\left(\frac{qx}{p}\right) - qx + p$$

If the initial populations correspond to the point A, ultimately the first species dies out and the second species increases in size to infinity. If the initial populations correspond to the point B, then ulitmately the second species dies out the first species tends to infinity. Similarly, if the initial populations correspond to point C, the first species dies out and the second species goes to infinity.

If the initial populations correspond to point E or F, the species populations converge to equilibrium populations $\frac{p}{q}, \frac{a}{b}$ and if the initial populations correspond to point G, H the first and second species die out respectively.

Thus except when the initial populations correspond to points on curves $O'E$ and $O'T$ only one species will survive in the competition process and the species can coexist only when the initial population sizes correspond to points on the curve EF.

It is also interesting to note that while the initial populations corresponding to A, E, B are quite close to one another, the ultimate behavior of these populations are drastically different. For populations starting at A, the second species alone survives, for populations strating at B, the first species alone survives, while population strating at E, both species can coexist. Thus a slight change in the initial population sizes can have a catastrophic effect on the ultimate behaviour.

It may also be noted that for both prey-predator and competition models, we have obtained a great deal of insight into the models without using the solution of Eqns. (1) , (2) or (7), (8). By using numerical methods of integration with the help of computers, we can draw some typical trajectories in both cases and can get additional insight into the behaviour of these models, $\frac{A_1}{C_2} > \frac{A_2}{B_2}$ and $\frac{A_1}{B_1} > \frac{A_2}{C_2}$.

Note:

1. The modification of the prey-predator model when

   - the predator population is harvested at a constant rate h_1.
   - the prey population is harvested at a constant rate h_2.
   - both species are harvested at constant rates.

2. The possibility of the existence of a stable age structure. (ie) age structure which does not change with time in the model.

MATHEMATICAL MODELING OF EPIDEMICS THROUGH SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS OF THE FIRST ORDER:

1. A Simple Epidemic Model:

Let $S(t)$ and $l(t)$ be the number of susceptibles (ie., those who can get a disease) and infected persons (ie., those who already have the disease).

Initially let there be n susceptible and one infected person in the system, so that

$$S(t) + I(t) = n + 1, S(0) = n, I(0) = 1$$

The number of infected persons grows at a rate proportional to the product of susceptible and infected persons and the number of susceptible persons decreases at the same rate so that we get the system of differential equations

$$\frac{dS}{dt} = -\beta SI, \frac{dI}{dt} = \beta SI$$

So that $\frac{dS}{dt} + \frac{dI}{dt} = 0, S(t) + I(t) = Constant = n + 1$

$$S + I = n + 1$$

$$I = n + 1 - S$$

$$S = n + 1 - I$$

and $\frac{dS}{dt} = -\beta S(n + 1 - S)$

$$\frac{dI}{dt} = -\beta I, (n + 1 - I)$$

$$\frac{dI}{I(n+1-I)} = \beta dt$$

consider, $\frac{1}{I(n+1-I)} = \frac{A}{I} + \frac{B}{n+1-I}$

$$1 = A(n + 1 - I) + BI$$

Put $I = 0$, $A = \frac{1}{n+1}$

Put $I = 1$, $B = 1 - \left(\frac{n}{n+1}\right)$

$$B = \frac{1}{n+1}$$

Integrating, $\int \frac{dt}{I(n+1-I)} = \int \beta \, dt$

$$\int \left(\frac{1}{(n+1)I} + \frac{1}{(n+1)(n+1-I)}\right) dI = \int \beta dt$$

$$\frac{1}{n+1}\int \left(\frac{1}{I} + \frac{1}{n+1-I}\right) dI = \beta t$$

$$\frac{1}{n+1}(\log I - \log(n+1-I)) + \log C = \beta t$$

$$\log\left(\frac{IC}{n+1-I}\right) = (n+1)\beta t$$

$$\frac{IC}{n+1-I} = e^{(n+1)\beta t} \qquad \text{--------------------- (5)}$$

Put $t = 0, I = 1$

$$\frac{C}{n} = e^0 \Longrightarrow c = n \qquad \text{--------------------- (6)}$$

Substitute (6) in (5), we get

$$\frac{In}{n+1-I} = e^{(n+1)\beta t}$$

$$In = (n+1-I)e^{(n+1)\beta t}$$

$$In = (n+1)e^{(n+1)\beta t} - Ie^{(n+1)\beta t}$$

$$In + Ie^{(n+1)\beta t} = (n+1)e^{(n+1)\beta t}$$

$$I\left(n + e^{(n+1)\beta t}\right) = (n+1)e^{(n+1)\beta t}$$

$$I = \frac{(n+1)e^{(n+1)\beta t}}{n+e^{(n+1)\beta t}} = \frac{(n+1)e^{(n+1)\beta t}.e^{-(n+1)\beta t}}{ne^{-(n+1)\beta t}+e^{(n+1)\beta t}e^{-(n+1)\beta t}}$$

$$I(t) = \frac{(n+1)}{ne^{-(n+1)\beta t}+1}$$

$t \to \infty, I = n+1$

$$I(t) + S(t) = n+1$$

$$S(t) = (n+1) - I(t)$$

$$= (n+1) - \frac{n+1}{ne^{-(n+1)\beta t}+1}$$

$$= (n+1)\left(1 - \frac{1}{ne^{-(n+1)\beta t}+1}\right)$$

$$= (n+1)\left(\frac{ne^{-(n+1)\beta t}+1-1}{ne^{-(n+1)\beta t}+1}\right)$$

$$S(t) = \frac{n(n+1)e^{-(n+1)\beta t}}{ne^{-(n+1)\beta t}+1}$$

so that, $\lim_{t\to\infty} S(t) = 0$ and $\lim_{t\to\infty} I(t) = (n+1)$ --------------------- (7)

2. A SUSCEPTIBLE –INFECTED-SUSCEPTIBLE (SIS) MODEL:

Here, a susceptible person can become infected at a rate proportional to SI and an infected person can recover and become susceptible again at a rate γ, so that

$$\frac{dS}{dt} = -\beta SI + \gamma I, \frac{dI}{dt} = \beta SI - \gamma I$$

Since, $S = n + 1 - I$ we get

$$\frac{dI}{dt} = \beta(n+1-I)I - \gamma I$$

$$= \beta(n+1) - \gamma I - \beta I^2$$

$$= I(\beta(n+1) - \gamma - \beta I)$$

$$\frac{dI}{I(\beta(n+1)-\gamma-\beta I)} = dt \qquad \text{----------------------- (8)}$$

Consider $\frac{1}{I(\beta(n+1)-\gamma-\beta I)} = \frac{A}{I} + \frac{B}{\beta(n+1)-\gamma-\beta I}$ ----------------------- (9)

$$1 = A(\beta(n+1) - \gamma - \beta I) + BI$$

put $I = 0$,we get $A = \frac{1}{\beta(n+1)-\gamma}$

put $I = 1$, we get $1 = \left(\frac{1}{\beta(n+1)-\gamma}\right)(\beta(n+1) - \gamma - \beta) + B$

$$1 = \frac{\beta n-\gamma}{\beta(n+1)-\gamma} + B$$

$$B = 1 - \frac{\beta n-\gamma}{\beta(n+1)-\gamma}$$

$$B = \frac{\beta(n+1)-\gamma-\beta n+\gamma}{\beta(n+1)-\gamma}$$

$$B = \frac{\beta}{\beta(n+1)-\gamma}$$

Substitute the value of A and B in (9), we get

$$\frac{1}{I(\beta(n+1)-\gamma-\beta I)} = \frac{\left(\frac{1}{\beta(n+1)-\gamma}\right)}{I} + \frac{\left(\frac{\beta}{\beta(n+1)-\gamma}\right)}{\beta(n+1)-\gamma-\beta I}$$

Integrating (8), we geet

$$\int \frac{dI}{I(\beta(n+1)-\gamma-\beta I)} = \int dt$$

$$\int \left(\frac{1}{I(\beta(n+1)-\gamma)} + \frac{\beta}{(\beta(n+1)-\gamma)(\beta(n+1)-\gamma-\beta I)}\right) dI \; = t$$

$$\frac{1}{\beta(n+1)-\gamma}\int\left(\frac{1}{I}+\frac{\beta}{\beta(n+1)-\gamma-\beta I}\right)dI = t$$

$$\frac{1}{\beta(n+1)-\gamma}\left(\log I+\beta\log(\beta(n+1)-\gamma-\beta I)\left(\frac{1}{-\beta}\right)\right) = t+\log c$$

$$\frac{1}{\beta(n+1)-\gamma}(\log I-\log(\beta(n+1)-\gamma-\beta I))-\log c = t$$

$$\log I-\log(\beta(n+1)-\gamma-\beta I)-\log c = t(\beta(n+1)-\gamma)$$

$$\log\left(\frac{I}{c(\beta(n+1)-\gamma-\beta I)}\right) = t(\beta(n+1)-\gamma)$$

$$\frac{I}{c(\beta(n+1)-\gamma-\beta I)} = e^{t(\beta(n+1)-\gamma)} \qquad \text{--------------------- (10)}$$

Apply the initial conditions $t = 0$ and $I = 1$, we get

$$\frac{1}{c(\beta(n+1)-\gamma-\beta)} = 1$$

$$c = \frac{1}{\beta n-\gamma}$$

substitute the c value in (10), we get

$$\frac{I(\beta n-\gamma)}{\beta(n+1)-\gamma-\beta I} = e^{t(\beta(n+1)-\gamma)}$$

$$I(\beta n-\gamma) = e^{t(\beta(n+1)-\gamma)}(\beta(n+1)-\gamma-\beta I)$$

$$I(\beta n-\gamma)+\beta I e^{t(\beta(n+1)-\gamma)} = e^{t(\beta(n+1)-\gamma)}(\beta(n+1)-\gamma)$$

$$I\left(\beta n-\gamma+\beta e^{t(\beta(n+1)-\gamma)}\right) = e^{t(\beta(n+1)-\gamma)}(\beta(n+1)-\gamma)$$

$$I = \frac{e^{t(\beta(n+1)-\gamma)}(\beta(n+1)-\gamma)}{\beta n-\gamma+\beta e^{t(\beta(n+1)-\gamma)}}$$

Multiply and Dividing by $e^{-t(\beta(n+1)-\gamma)}$, we get

$$I = \frac{(\beta(n+1)-\gamma)}{(\beta n-\gamma)e^{-t(\beta(n+1)-\gamma)}+\beta}$$

when $t \to \infty$, $I(t) \to \beta(n+1) - \gamma$

$$I(t) \to (n+1) - \frac{\gamma}{\beta}$$

$$I(t) \to (n+1) - \rho, \text{ where } \rho = \frac{\gamma}{\beta}$$

Since $S(t) + I(t) = n + 1$

$$S(t) = (n+1) - I(t)$$

$$S(t) = (n+1) - (n+1) + \rho$$

$$S(t) = \rho$$

3. SIS Model with Constant Number of Carriers:

Here infection is spread both by infectives and a constant number C of carriers, so that Eqn. (7) becomes

$$\frac{dI}{dt} = (\beta(I+C)S) - \gamma I$$

$$= \beta(I+C)(n+1-I) - \gamma I$$

$$= \beta(I+C)(n+1) - \beta I(I+C) - \gamma I$$

$$= \beta I(n+1) + \beta C(n+1) - \beta I^2 - \beta IC - \gamma I$$

$$= I(\beta(n+1) - \beta C - \gamma) + \beta C(n+1) - \beta I^2$$

$$= \beta C(n+1) + \beta\left(n+1-C-\frac{\gamma}{\beta}\right) I - \beta I^2 \qquad \text{-------------------- (11)}$$

4. Simple Epidence Model with Carriers:

In this model, only carriers spread the disease and their number decreases exponentially with time as these are identified and eliminated, so that we get

$$\frac{ds}{dt} = -\beta S(t)C(t) + \gamma I(t) \qquad \text{-------------------- (12)}$$

$$\frac{dI}{dt} = \beta C(t)S(t) - \gamma I(t) \qquad \text{-------------------- (13)}$$

$$\frac{dC}{dt} = -\alpha C$$

Integrating, $\int \frac{dC}{C} = -\alpha \int dt$

$$\log C = -\alpha t + a$$

$$C = e^{-\alpha t + a}$$

$$C = e^{-\alpha t}.e^{a}$$

$$C(t) = c_0 e^{-\alpha t}$$

Since, $S(t) + I(t) = N$

From (13), $\frac{dI}{dt} = \beta c_0 e^{-\alpha t}\big(N - I(t)\big) - \gamma I(t)$

$$= \beta c_0 N e^{-\alpha t} - \beta c_0 e^{-\alpha t} I(t) - \gamma I(t)$$

$$= \beta c_0 N e^{-\alpha t} - (\beta c_0 e^{-\alpha t} - \gamma) I(t) \qquad \text{------------------ (14)}$$

5. Model with Removal:

Here infected persons are removed by death or hospitalisation at a rate proportional to the number of infectives, so that the model is

$$\frac{dS}{dt} = -\beta SI$$

$$\frac{dI}{dt} = \beta SI - \gamma I = \beta I \left(S - \frac{\gamma}{\beta}\right)$$

Since $\rho = \frac{\gamma}{\beta}$, we get $\frac{dI}{dt} = \beta I(S - \rho)$

with initial ccondtions $S(0) = S_0 > 0, I(0) = I_0 > 0, R(0) = R_0 > 0$

$$S_0 + I_0 = N$$

6. Model with Removal and Immigration:

We modify the above model to allow for the increase of susceptibles at a constant rate μ so that the model is

$$\frac{dS}{dt} = -\beta SI + \mu$$

$$\frac{dI}{dt} = \beta SI - \gamma I$$

$$\frac{dR}{dt} = \gamma I$$

MATHEMATICAL MODELS IN MEDICINE, ARMS RACE BATTLES AND INTERNATIONAL TRADE IN TERMS OF SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS:

A Model for Diabetes Mellitus:

Let $x(t), y(t)$ be the blood sugar and insulin levels in the blood stream at time t. The rate of change $\frac{dy}{dt}$ of insulin level is proportional to

(i) the excess $x(t) - x_0$ of sugar in blood over its fasting level, since this excess makes the pancreas secrete insulin into the blood stream.

(ii) the amount $y(t)$ of insulin since insulin left to itself tends to decay at a rate proportional to its amount.

(iii) the insulin dose $d(t)$ injected per unit time. This give

$$\frac{dy}{dt} = a_1(x - x_0)H(x - x_0) - a_2 y + a_3 d(t) \qquad \text{------------------ (15)}$$

where a_1, a_2, a_3 are positive constants and $H(x)$ is a step function which takes the value unity when $x > 0$ and taken the value zero otherwise. This occurs in (15) because if blood sugar level is less than x_0, there is no secretion of insulin from the pancreas.

Again the rate of change $\frac{dx}{dt}$ of sugar level is proportional to

(i) the product xy since the higher the levels of sugar and insulin, the higher is the metabolism of sugar.

(ii) $x_0 - x$ since if sugar level falls below fasting level, sugar is released from the level stores to raise the sugar level to normal.

(iii) $x - x_0$ since if $x > x_0$, there is a natural decay in sugar level proportional to its excess over fasting level.

(iv) The function of $t - t_0$ where t_0 is the time at which food is taken

$$\frac{dx}{dt} = -b_1 xy + b_2(x_0 - x)H(x_0 - x) - b_3(x - x_0)H(x - x_0) + b_4 z(t - t_0) \text{ --- (16)}$$

where a suitable form for $z(t - t_0)$ can be

$$z(t - t_0) = 0, t < t_0$$

$$= Qe^{-\alpha(t - t_0)}, t > t_0$$

Eqn. (15) and (16) give two simultaneous differential equations to determine $x(t)$ and $y(t)$. These equation can be numerically integrated.

UNIT –IV

MATHEMATICAL MODELLING THROUGH DIFFERENCE EQUATIONS

Introduction:

Mathematical modeling using difference equations is the process of representing and analyzing systems that change over discrete time intervals, such as daily or monthly steps. These equations are the discrete-time equivalent of differential equations and are used across various fields like biology, economics, and finance to model phenomena where quantities change in distinct steps rather than continuously. Examples include tracking a population's growth from one year to the next or calculating a bank account's balance with monthly interest.

The Need For Mathematical Modeling Through Difference Equations:

1. Simple Models:

We need difference equation models when either the independent variable is discrete or it is mathematically convenient to treat it as a discrete variable.

Thus is Genetics, the genetic characteristics change from generation to generation and the variable representing a generation is a discrete variable.

In Economics, the price changes are considered from year to year or from month to month or from week to week or from day to day. In every case, the time variable is discretized.

In Population Dynamics, we consider the changes in population from one age-group to another and the variable representing the age-group is a discrete variable.

In finding the probability of n persons in a queue or the probability of n persons in a state or the probability of n successes in a certain number of trials, the independent variables is discrete.

For mathematical modeling through differential equations, we give an increment Δx to independent variable x, find the change Δy in y and let $\Delta x \to 0$ to get differential equations. In most cases, we cannot justify the limiting process rigorously. Thus for modeling fluid motion, making $\Delta x \to 0$ has no meaning since a fluid consists of a large number of particles and the distance between two neighbouring particles cannot be made arbitrary small. Continuum mechanics is only an approximation (through fortunately a very good one) to reality.

Even if the limiting process can be justified. (eg) when the independent variable is time, the resulting differential equation may not be solvable analytically. We then solve it numerically and for this purpose, we again replace the differential equation by a system of difference equations. Numerical methods of solving differential equations essentially mean solving difference equations.

It is even argued that since in most cases, we have to ultimately solve difference equations, we may avoid modeling through differential equations altogether. This is of course going too far since as we have seen in earlier chapters, mathematical modeling through differential equations is of immense importance to science and technology. Another argument in favour of difference equation models is that those biological and social scientists who do not know calculus and transcendental numbers like e can still work with difference equation models and some important consequences of these models can be deduced with the help of even pocket calculators by even high school students.

We now give simple difference equation models parallel to the differential equation models studied in earliear chapters.

(i) Population Growth Model: If the population at time t is $x(t)$, then assuming that the number of births and deaths in the next unit interval of time are proportional to the populations at time t, we get the model:

$$x(t+1) - x(t) = bx(t) - dx(t)$$

$$x(t+1) = ax(t) \qquad \text{--------------------- (1)}$$

so that $x(t) = ax(t-1) = a^2x(t-2)$

$$= a^2x(t-2) = a^3x(t-3) = \cdots = a^t x(0) \qquad \text{---------------------- (2)}$$

This may be compared with the differential equation model:

$$\frac{dx}{dt} = ax \text{ with the solution } x(t) = x(0)e^{at} \qquad \text{---------------------- (3)}$$

For solving the difference equation model, we require only simple algebra, but for solving the differential equation model, we require knowledge of calculus, differential equation and exponential functions.

(ii) Logistic Growth Model:

This is given by $x(t+1) - x(t) = ax(t) - bx^2(t)$ ---------------------- (4)

This is not easy to solve, but given $x(0)$, we can find $x(1), x(2), x(3), \ldots$ in succession and we can get a fairly good idea of the behaviour of the model with the help of a pocket calculator.

(iii) Prey-Predator Model:

This is given by

$$x(t+1) - x(t) = -ax(t) + bx(t)y(t) \qquad a, b > 0$$

$$y(t+1) - y(t) = py(t) - qx(t)y(t) \qquad p, q > 0$$

and again given $x(0), y(0)$, we can find $x(1), y(1); x(2), y(2); x(3), y(3), \ldots$ in succession.

(iv) Competition Model:

This is given by

$$x(t+1) - x(t) = ax(t) - bx(t)y(t) \qquad a, b > 0$$

$$y(t+1) - y(t) = py(t) - qx(t)y(t) \qquad p, q > 0$$

(v) Simple Epidemics Model:

This is given by

$$\left.\begin{aligned} x(t+1)-x(t) &= -\beta x(t)y(t) \\ y(t+1)-y(t) &= \beta x(t)y(t) \end{aligned}\right\}, \ \beta > 0$$

PROBLEM

1. For the population growth model, let $\boldsymbol{x(0) = 100}$, $\boldsymbol{a = 0.5}$ or $\boldsymbol{1}$ or $\boldsymbol{2}$. Find $\boldsymbol{x(t)}$ for $\boldsymbol{t = 1}$ to $\boldsymbol{50}$ and plot $\boldsymbol{x(t)}$ as a function of $\boldsymbol{t}$ in each case.

Solution:

Let us assumed exponential growth model is given by the equation

$$x(t) = x(0)e^{at} \qquad \text{------------------- (1)}$$

Given, the initial condition $x(0) = 100$

To find $\boldsymbol{x(t)}$:

For $t = 1$ to 50

Case (i): When $a = 0.5$, the function $x(t) = 100.e^{0.5t}$

Put $t = 1$, then $x(1) = 100.e^{0.5(1)} \approx 164.87$

Put $t = 2$, then $x(2) = 100.e^{0.5(2)} \approx 271.83$

Put $t = 3$, then $x(3) = 100.e^{0.5(3)} \approx 448.17$

… …

The plot $x(t)$ versus t will show a curve that increases at an acceleration rate, representing exponential growth.

Case (ii): When $a = 1$, the function $x(t) = 100.e^{1.t}$

Put $t = 1$, then $x(1) = 100.e^{1(1)} \approx 271.83$

Put $t = 2$, then $x(2) = 100.e^{1(2)} \approx 738.91$

Put $t = 3$, then $x(3) = 100.e^{1(3)} \approx 2008.55$

… …

The plot $x(t)$ versus t will also be a curve to showing exponential growth, but it will increase more rapidly than the case where $a = 0.5$.

Case (iii): When $a = 2$, the function $x(t) = 100.e^{2.t}$

Put $t = 1$, then $x(1) = 100.e^{2(1)} \approx 738.91$

Put $t = 2$, then $x(2) = 100.e^{2(2)} \approx 5459.82$

Put $t = 3$, then $x(3) = 100.e^{2(3)} \approx 40342.88$

… …

Therefore, the plot $x(t)$ versus t will show the steepest exponential growth curve among the abve cases.

2. For logistic growth model, let $x(0) = 100, a = 0.1$ and $b = 0.001$. Find $x(t)$ for $t = 1$ to 100 and plot $x(t)$ as a function of t.

Solution:

Let us assuming the logistic differential equation

$$\frac{dx}{dt} = ax - bx^2$$

and the initial condition is $x(0) = x_0$, the solution is given by

$$x(t) = \frac{ax_0}{bx_0 + (a - bx_0)e^{-at}}$$

Given: $x(0) = 100, a = 0.1$ and $b = 0.001$.

$$x(t) = \frac{0.1(100)}{0.001(100) + \big(0.1 - 0.001(100)\big)e^{-0.1(t)}}$$

$$= \frac{10}{0.1 + (0.1 - 0.1)e^{-0.1(t)}}$$

$$= \frac{10}{0.1}$$

$$x(t) = 100$$

Therefore, the value of $x(t) = 100$ for all value of t. A plot of $x(t)$ as a function of t from $t = 1$ to 100 would be a horizontal line at $y = 100$.

3. For prey-predator and competition model, let $\boldsymbol{x(0) = 40, y(0) = 10, a = 0.01}$, $\boldsymbol{b = 0.001, p = 0.005, q = 0.0001}$. Plot points $\boldsymbol{x(t), y(t)}$ for $\boldsymbol{t = 0}$ to $\boldsymbol{50}$.

Solution:

Let us assuming the prey-predator model with x representing the prey population and y representing the predator population. To calculate and polt the points for $x(t)$ and $y(t)$ for $t = 0$ to $t = 50$.

Prey-predator model:

W.K.T the Lotka-Volterra equations are given by:

$$\frac{dx}{dt} = ax - bxy$$

$$\frac{dy}{dt} = -py + qxy$$

Given the initial conditions $x(0) = 40, y(0) = 10$ and the parameters $a = 0.01$, $b = 0.001$, $p = 0.005$ and $q = 0.0001$, the equations become

$$\frac{dx}{dt} = 0.01x - 0.001xy$$

$$\frac{dy}{dt} = -0.005y + 0.0001xy$$

The solution to those differential equations can be approximated using numerical method. The following table provides the approximate value of $x(t)$ and $y(t)$ for t from 0 to 50.

t	$x(t)$	$y(t)$
0	40	10
5	39.06	10.15
10	37.19	10.37
15	34.61	10.66
20	31.57	11.05
25	28.39	11.53

30	25.46	12.10
35	22.99	12.72
40	21.05	13.31
45	19.64	13.78
50	18.71	14.07

Competition model:

The Lotka-Volterra competition model describes the population dynamics of two species that compete for the same resources. The system of differential equations for this model is given by

$$\frac{dx}{dt} = x(a - bx - py)$$

$$\frac{dy}{dt} = y(q - by - px)$$

where $x(t)$ and $y(t)$ are the population of the two species at time t. The given parameters are $x(0) = 40, y(0) = 10$ and the parameters $a = 0.01,$ $b = 0.001,$ $p = 0.005$ and $q = 0.0001.$

$$\frac{dx}{dt} = x(0.01 - 0.001x - 0.005y)$$

$$\frac{dy}{dt} = y(0.0001 - 0.001y - 0.005x)$$

To polt the points $x(t)$ and $y(t)$ for $t = 0$ to $t = 50$, you would numerically solve this system of differential equations with the initial conditions $x(0) = 40$ and $y(0) = 10$.

Therefore, the resulting plot would show the population trajectories of both species over time.

4. For Simple Epidemics model, let $x(0) = 100, y(0) = 1, \beta = 0.5$, plot $x(t), y(t)$ in the xy-plane for $t = 0$ to 100.

Solution:

The Simple Epidemics model, also known as the Susceptible-Infectious (SI) model, is described by a system of differential equations that track the susceptible population, x and the infectious population$y$. The equations are

$$\frac{dx}{dt} = -\beta xy$$

$$\frac{dy}{dt} = \beta xy$$

The user provides the following initial conditions and parameter $x(0) = 100$, $y(0) = 1$ and $\beta = 0.5$. The total population is the sum of the susceptible and infectious population, which is constant $N = x(t) + y(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\beta xy}{\beta xy} = -1$$

Integrating, $\int dy = -\int dx + C$

$$y(t) = -x(t) + C \qquad \text{---------------- (1)}$$

Using initial conditions $t = 0$, we get

$$y(0) = -x(0) + C$$

$$1 = -100 + C$$

$$C = 101$$

From (1), $y = -x + 101$

Put $x = 0$, then $y = 101$

Put $x = 100$, then $y = 1$

Therefore, the trajectory starts at the initial point $(100, 1)$ and moves along the line towards the point $(0, 101)$ as time progresses. The time interval from $t = 0$ to $t = 100$ maps a specific portion of this line segment.

BASIC THEORY OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS:

This theory is parallel to the corresponding theory of linear differential equations with constant coefficients, but is not usually taught in many places.

1. The Linear Difference Equation:

An equation of the form

$$f(x_{t+n}, x_{t+n-1}, \ldots, x_t, t) = 0 \qquad \text{---------------- (5)}$$

is called a difference equations of n^{th} order. The equation

$$f_0(t)x_{t+n} + f_1(t)x_{t+n-1} + \cdots + f_n(t)x_t = \phi(t) \qquad \text{--------------- (6)}$$

is called a linear difference equation, since it involves $x_t, x_{t+1}, \dots, x_{t+n}$ only in the first degree. The equation

$$a_0 x_{t+n} + a_1 x_{t+n-1} + \dots + a_n x_t = \phi(t) \qquad \text{----------------- (7)}$$

is called a linear difference equation with constant coefficients. The equation

$$a_0 x_{t+n} + a_1 x_{t+n-1} + \dots + a_n x_t = 0 \qquad \text{---------------- (8)}$$

is called homogeneous linear difference equations with constant coefficients. Let $x_t = g_1(t), g_2(t), \dots, g_n(t)$ be n linearly independent solutions of (8), then

$$x_t = A_1 g_1(t) + A_2 g_2(t) + \dots + A_n g_n(t) \qquad \text{---------------- (9)}$$

is also a solution of (8) where $A_1, A_2, \dots, A_n$ are n arbitrary constants. This is the ost general solution of (8).

Again it can be shown that if $G_1(t)$ is the solution of (8) containing n arbitrary constants and $G_2(t)$is any particular solution of (7) containing no arbitrary constant, then $G_1(t) + G_2(t)$ is the most general solution of (7), $G_1(t)$ is called the complementary function and $G_2(t)$ is called a particular solution.

2. The Complementary Function:

We try the solution $x_t = a\lambda^t$. If this satisfies (8) , we get

$$g(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_n = 0 \qquad \text{----------------- (10)}$$

This algebraic equation of n^{th} degree has n roots $\lambda_1, \lambda_2, \dots, \lambda_n$ real or complex. The complementary function is then given by

$$G_1(t) = c_1\lambda_1^t + c_2\lambda_2^t + \cdots + c_n\lambda_n^t \qquad \text{---------------- (11)}$$

Case (i): If $\lambda_1, \lambda_2, \dots, \lambda_n$ are all real and distinct, (11) gives us the complementary function when $c_1, c_2, \dots, c_n$ are any n arbitrary real constants.

Case (ii): If two of the roots λ_1, λ_2 are equal, then (11) contains only $n-1$ arbitrary constants and as such it cannot be the most general solution..

we try the solution $ct\lambda_1^t$, we get

$$a_0(t+n)\lambda_1^n + a_1(t+n-1)\lambda_1^{n-1} + \cdots + a_n = 0$$

or $$tg(\lambda_1) + g'(\lambda_1) = 0 \qquad \text{---------------- (12)}$$

which is identically satisfied since both $g(\lambda_1) = 0$ and $g'(\lambda_1) = 0$ as λ_1 is an repeated root. In this case

$$G_1(t) = (c_1 + c_2 t)\lambda_1^t + c_3\lambda_3^t + c_4\lambda_4^t + \cdots + c_n\lambda_n^t \qquad \text{---------------- (13)}$$

Case(iii): If a root λ_t is repeated k times, the complementary function is

$$G_t(t) = (c_1 + c_2 t + c_3 t^2 + \cdots + c_k t^{k-1})\lambda_1^t + c_{k+1}^t$$

$$+ \cdots + c_n\lambda_n^t \qquad \text{---------------- (14)}$$

Case (iv): Let $g(\lambda) = 0$ have two complex roots $\alpha \pm i\beta$, then their contribution to complementary function is

$$c_1(\alpha + i\beta)^t + c_2(\alpha - i\beta)^t \qquad \text{---------------- (15)}$$

Putting $\alpha = r\cos\theta, \beta = r\sin\theta$ and using De Moivre's theorem, this reduces to

$$c_1 r^t(\cos\theta + i\sin\theta)^t + c_2 r^t(\cos\theta - i\sin\theta)^t$$

$$= r^t\cos(\theta t)(c_1 + c_2) + r^t\sin(\theta t)(ic_1 - ic_2)$$

$$= r^t(d_1\cos(\theta t) + d_2\sin(\theta t))$$

$$= (\alpha^2 + \beta^2)^{\frac{t}{2}}(d_1\cos(\theta t) + d_2\sin(\theta t)) \quad \text{------------------ (16)}$$

where $\tan\theta = \frac{\beta}{\alpha}$ and d_1, d_2 are arbitrary constants ------------------ (17)

Case (v): If the complex roots $\alpha + i\beta$ are repeated k times, then contribution to the complementary function is

$$(\alpha^2 + \beta^2)^{\frac{t}{2}}((d_0 + d_1 t + \cdots + d_{k-1}t^{k-1}cos\ (\dot{\theta} t)$$

$$+(f_0 + f_1 t + \cdots + f_{k-1}t^{k-1})\sin(\theta t) \quad \text{------------------ (18)}$$

where $d_0, d_1, \dots, d_{k-1}, f_0, f_1, \dots, f_{k-1}$ are $2k$ arbitrary constants.

3. The Particular Solution:

Here we want a solution of (7) not containing any arbitrary constant.

Case (i) Let $\phi(t) = AB^t$, B is not a root of $g(\lambda) = 0$ ----------------- (19)

we try the solution CB^t, substituting in (7), we get

$$CB^t(a_0B^n + a_1B^{n-1} + \cdots + a_n) = AB^t \quad \text{----------------- (20)}$$

If $B \neq \lambda_1, \lambda_2, \dots, \lambda_n$ we get

$$C = \frac{A}{a_0 B^n + a_1 B^{n-1} + \cdots + a_n} \quad \text{----------------- (21)}$$

and the particular solution is

$$\frac{AB^t}{a_0 B^n + a_1 B^{n-1} + \cdots + a_n} \quad \text{----------------- (22)}$$

Case (ii): Let $\phi(t) = AB^t, B$ is a non-repeated root of $g(\lambda) = 0$ ----------------- (23)

we try the solution CtB^t. substituting in (7), we get

$$B^t\big(Ctg(B) + Cg'(B)\big) = AB^t \quad \text{----------------- (24)}$$

Since $g(B) = 0, g'(B) \neq 0$

$$C = \frac{A}{g'(B)} \quad \text{----------------- (25)}$$

so that the particular solution is

$$\frac{AtB^t}{a_0 n B^{n-1} + a_1 (n-1) B^{n-2} + \cdots + a_{n-1}} \quad \text{------------------ (26)}$$

Case (iii): Let $\phi(t) = AB^t, g(B) = 0, g'(B) = 0, \ldots, g^{k-1}(B) = 0$ ----------------- (27)

where $g^k(B) \neq 0$

then the particular solutionn is $\frac{At^{k-1}B^t}{g^k(B)}$ ----------------- (28)

Case (iv): Let $\phi(t) = At^k$ ----------------- (29)

we try the solution $d_0 t^k + d_1 t^{k-1} + d_2 t^{k-2} + \cdots + d_k$ ----------------- (30)

Substituting in (7), we get

$$a_0(d_0(t+n)^k + d_1(t+n)^{k-1} + d_2(t+n)^{k-2} + \cdots + d_k)$$

$$+a_1(d_0(t+n-1) + d_1(t+n-1)^{k-1} + d_2(t+n-1)^{k-2}$$

$$+\cdots + d_k) + \cdots + a_n(d_0t^k + d_1t^{k-1} + d_2t^{k-2} + \cdots + d_k) = 0$$

--------------- (31)

Equating the coefficients of $t^k, t^{k-1}, \dots, t^0$ on both sides, we get $k+1$ equations which in general will enables us to determine $d_0, d_1, d_2, \dots, d_k$ and thus the particular solution will be determined.

UNIT – V

MATHEMATICAL MODELLING THROUGH DIFFERENCE EQUATIONS IN ECONOMICS AND FINANCE

INTRODUCTION:

Mathematical modeling through difference equations provides a powerful and discrete framework for analyzing dynamic systems in economics and finance. Unlike differential equations, which model continuous change, difference equations describe systems that evolve in discrete time steps, such as annually, quarterly, or daily, perfectly reflecting the nature of many financial and economic processes.

In economics, difference equations are fundamental to understanding phenomena like:

- **Business Cycles:** Modelling how economic output fluctuates over time due to interactions between consumption, investment, and government spending.
- **Supply and Demand Dynamics:** Analyzing how prices and quantities adjust in markets over successive periods.
- **Growth Models:** Describing the period-by-period accumulation of capital and national income.

In finance, these models are essential for:

- **Asset Pricing:** Calculating the value of financial instruments, like bonds or stocks, over discrete time periods.
- **Risk Management:** Modeling the discrete-time evolution of asset prices and volatility.

- **Option Valuation:** Analyzing how the value of derivatives changes at specific intervals.

1. The Harrod Model:

Let $S(t), \gamma(t), I(t)$ denote the savinga, national income and investment respectively. We make now the following assumptions:

(i) Savings made by the people in a country depends on the national income.

(ie) $S(t) = \alpha\, Y(t), \quad \alpha > 0$ ------------------ (1)

(ii) The investment depends on the difference between the income of the current year and the last year.

(ie) $I(t) = \beta\, \big(Y(t) - Y(t-1)\big), \quad \beta > 0$ ------------------ (2)

(iii) All the savings made are invested, so that

$S(t) = I(t)$ ------------------ (3)

From (1), (2) and (3), we get the difference equation

$Y(t) = \frac{\beta}{\beta-\alpha}\, Y(t-1)$ ------------------ (4)

which has the solution

$Y(t) = A\left(\frac{\beta}{\beta-\alpha}\right)^t = Y(0)\left(\frac{\beta}{\beta-\alpha}\right)^t$ ------------------ (5)

Assuming that $Y(t)$ is always positive,

$\beta > \alpha, \frac{\beta}{\beta - \alpha} > 1$ ----------------- (6)

so that the national incomme increases with t. The national incomes at different times $0, 1, 2, 3, \ldots$ from a geometrical progression.

Thus, if all savings are invested, savings are proportional to national income and the investment is proportional to the excess of the current years income over the preceding years income, then the national income increases geometrically.

2. The Cobweb Model:

Let p_t = price of a commodity in the year t and

q_t = amount of the commodity available in the market in year t, then we make the following assumptions.

(i) Amount of the commodity produced this year and available for sale is a linear function of the price of the commodity in the last year.

(ie) $q_t = \alpha + \beta p_{t-1}$ ------------------ (7)

whare $\beta > 0$ since if the last year's price was high, the amount available this year would also be high.

(ii) The price of the commodity this year is a linear function of the amount available this year.

(ie) $p_t = \gamma + \delta q_t$ ----------------- (8)

where $\delta < 0$, since is q_t is large, the price would be low. From (7) and (8)

$$p_t - \beta\delta p_{t-1} = \gamma + \alpha\delta \qquad \text{----------------- (9)}$$

which has the solution

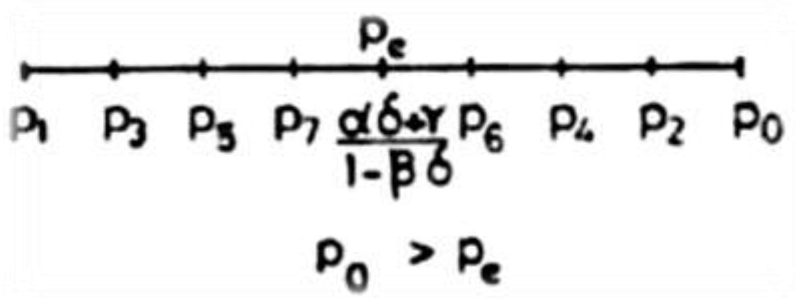
$$\left(p_t - \frac{\alpha\delta+\gamma}{1-\beta\delta}\right) = \left(p_0 - \frac{\alpha\delta+\gamma}{1-\beta\delta}\right)(\beta\delta)^t \qquad \text{----------------- (10)}$$

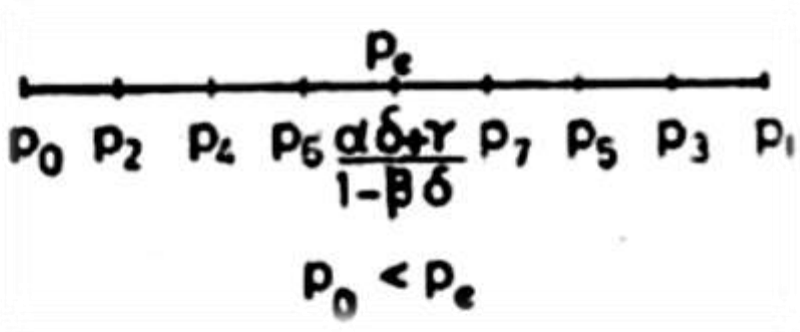
So that $\left(p_t - \frac{\alpha\delta+\gamma}{1-\beta\delta}\right) = \left(p_{t-1} - \frac{\alpha\delta+\gamma}{1-\beta\delta}\right)(\beta\delta)$ ----------------- (11)

Since $\beta\delta$ is negative $p_0, p_1, p_2, p_3, \ldots$ are alternatively greater and less than $\frac{\alpha\delta+\gamma}{1-\beta\delta}$.

If $|\beta\delta| > 1$, the deviation of p_t from $\frac{\alpha\delta+\gamma}{1-\beta\delta}$ goes on increasing. On the other hand if $|\beta\delta| < 1$, this deviation goes on decreasing and ultimately $p_t \to \frac{\alpha\delta+\gamma}{1-\beta\delta}$ as $t \to \infty$.

Shown below in the figure, how the price approaches the equilibrium price $p_t = \frac{\alpha\delta+\gamma}{1-\beta\delta}$ as t increases in the two cases when $p_0 > p_e$ and $p_0 < p_e$ respectively.





In the same way, eliminating p_t from (6), (7) we get

$$q_t = \alpha + \beta\gamma + \beta\delta q_{t-1} \qquad \text{-------------------- (12)}$$

which has the solution

$$\left(q_t - \frac{\alpha+\beta\gamma}{1-\beta\delta}\right) = \left(q_e - \frac{\alpha+\beta\gamma}{1-\beta\delta}\right)(\beta\delta)^t \qquad \text{------------------- (13)}$$

so that q_t also oscillates about the equilibrium quantity level

$$q_t = \frac{\alpha+\beta\gamma}{1-\beta\delta} \text{ if } |\beta\delta| < 1$$

The variuation of both prices and quantities is shown simultaneously in the below figure.

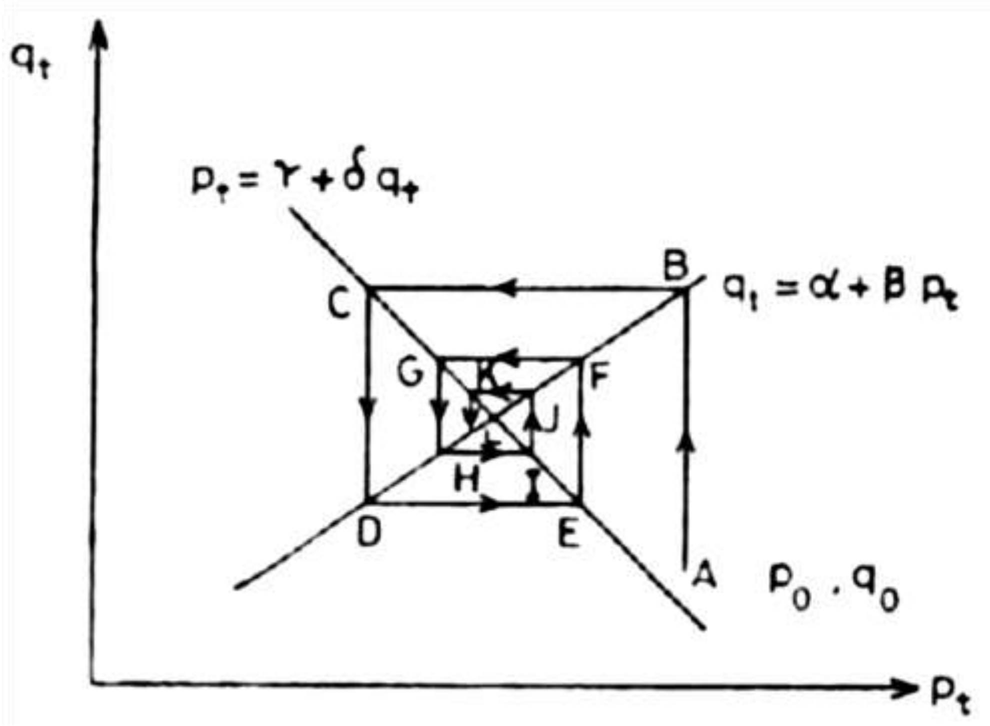
Suppose we start in the year zero with price p_0 and quantity q_0 represented by the point A. In year 1, the quantity q_1 is given by $\alpha + \beta p_0$ and the price is given by $p_1 = \gamma + \delta q_t$. This brings us to the point C in two steps via B. The path of prices and quantities is thus given by the Cobweb path $ABCDEFGHI$, ... and the equilibrium price and quantity are given by the intersection of the two straight lines.

3. Samuelson's Interaction Models:

The basic equations for the first interaction model are

$$Y(t) = C(t) + I(t), C(t) = \alpha Y(t-1), I(t) = \beta\big(C(t) - C(t-1)\big) \text{ --------- } (14)$$

Here the positive constant α is the marginal propensity to sonsume with respect to income of the previous year and the positive constant β is the



relation given by the acceleration principle. (ie) β is the increase in investment per unit of excess of this years consumption over the last year's.

From (14), we get the scond order difference equation

$$Y(t) - \alpha(1+\beta)Y(t-1) + \alpha\beta Y(t-2) = 0 \qquad \text{------------------ } (15)$$

In the second interaction model, there is an additional investment by the government and this investment is assumed to be a constant γ. In this case (15) is modified to

$$Y(t) - \alpha(1+\beta)Y(t-1) + \alpha\beta Y(t-2) - \gamma = 0 \qquad \text{-----------------} (16)$$

The solution of (15) and (77) can show either an increasing trend in $Y(t)$ or a decreasing trend in $Y(t)$ or an oscillating trend in it.

4. Application to Acturial Science:

One important aspect of actuarial science is what is called mathematics of finance or mathematics of investment.

If a sum S_0 is invested at compound interest of i per unit amount per unit time and S_t is the amount at the end of time t, then we get the difference equation

$$s_{t+1} = S_t + iS_t = (1+i)S_t \qquad \text{-----------------} (17)$$

which has the solution

$$S_t = S_0(1+i)^t \qquad \text{-----------------} (18)$$

which is the well-known formula for compound interest.

Suppose a person borrows a sum S_0 at compound interest i and wants to amortize his debt, ie he wants to pay the amount and interest back by payment of n equal instalments, say R, the first payment to be made at the end of the first year.

Let S_t be the amount due at the end of t years, then we have the differencee equation

$$S_{t+1} = S_t + iS_t - R = (1+i)S_t - R \qquad \text{----------------- (19)}$$

It solution is

$$S_t = \left(S_0 - \frac{R}{i}\right)(1+i)^t + \frac{R}{i}$$

$$= S_0(1+t)^t - R\frac{(1+i)^t - 1}{i} \qquad \text{------------------ (20)}$$

If the amount is paid back in n years, $S_n = 0$ so that

$$R = S_0 \frac{i}{1-(1+i)^{-n}} = S_0 \frac{1}{\bar{n}|i} \qquad \text{------------------ (21)}$$

where $a_{\bar{n}|i}$ called the amortization factor is the present value of an annuity of 1 per unit time for n periods at an interest rate i.

The functions $a_{\bar{n}|i}$ and $\left(a_{\bar{n}|i}\right)^{-1}$ are tabulated for common values of n and i.

Suppose an amount R is deposited at the end of every period in an bank and let S_i be the amount at the end of t periods, then

$$S_{t+1} = S_t(1+i) + R \qquad \text{---------------- (22)}$$

so that (since $S_0 = 0$)

$$S_n = R\frac{(1+i)^n - 1}{i} = R\, S_{\bar{n}|i} \qquad \text{---------------- (23)}$$

From (21) and (23), we get

$$S_{\bar{n}|i} = (1+i)^n a_{\bar{n}|i} \qquad \text{--------------- (24)}$$

or $\frac{1}{s_{\bar{n}|i}} = \frac{(1+i)^{-n}}{a_{\bar{n}|i}}$ --------------- (25)

If a person has to pay an amount S at the end of n years, he can do it by paying into a sinking find an amount R per period where

$$R = S\frac{1}{s_{\bar{n}|i}} \quad \text{--------------- (26)}$$

where $\frac{1}{s_{\bar{n}|i}}$ is the sinking fund factor and can be tabulated by using (25).